Feedback Stabilization of a Fluid–Rigid body Interaction System

Mehdi Badra[*], Takéo Takahashi [†‡]

June 16, 2014

Abstract

We study the feedback stabilization of a system composed by an incompressible viscous fluid and a rigid body. We stabilize the position and the velocity of the rigid body and the velocity of the fluid around a stationary state by means of a Dirichlet control, localized on the exterior boundary of the fluid domain and with values in a finite dimensional space. Our first result concerns weak solutions in the two-dimensional case, for initial data close to the stationary state. Our method is based on general arguments for stabilization of nonlinear parabolic systems combined with a change of variables to handle the fact that the fluid domain of the stationary state and of the stabilized solution are different. This additional difficulty leads to the assumption that the initial position of the rigid body is the position associated to the stationary state. Without this hypothesis, we work with strong solutions, and to deal with compatibility conditions at the initial time, we use finite dimensional dynamical controls. We prove again that for initial data close to the stationary state, we can stabilize the position and the velocity of the rigid body and the velocity of the fluid. In the three dimensional case, we also obtain the local stabilization of strong solutions with finite dimensional dynamical controls.

Mathematics Subject Classification (2010): 74F10, 35Q35, 76D55, 76D05, 93D15.

Key words: feedback stabilization, fluid-structure interaction, Navier-Stokes equations.

1 Introduction and main results

The main goal of the present work is to prove the stabilizability of a fluid-structure system describing the motion of a solid immersed in a viscous fluid. It follows the previous study of the authors [6] about a simplified 1D fluid particle system. Here, the fluid is described by the Navier-Stokes equations, the equations of the rigid body are obtained by applying Newton's laws and the control takes the form of a Dirichlet condition, localized on the exterior boundary and with values in a finite dimensional space. Moreover, the target state to be stabilized is stationary but, as in [6] and unlike other connected works on the subject, it involves a steady velocity which is not assumed to be zero. A precise description of the system is given below.

Let Ω be a bounded smooth domain (say, of class $C^{2,1}$) of $\mathbb{R}^d$, $d = 2$ or $d = 3$ that contains both the fluid and the structure. We consider a rigid body of shape $\mathcal{S}$ moving inside Ω. We assume that $\mathcal{S}$ is a smooth (say, of class $C^{2,1}$), connected and compact subset of $\mathbb{R}^d$ with non empty interior.

For all $h \in \mathbb{R}^d$ and for all $R \in SO(d)$ (the special orthogonal group of $\mathbb{R}^d$), we set

$$\mathcal{S}(h,R) = h + R\mathcal{S}, \quad \mathcal{F}(h,R) = \Omega \setminus \mathcal{S}(h,R).$$

We are interested in the admissible positions of the rigid body, i.e. in couples (h,R) such that $\mathcal{S}(h,R) \subset \Omega$. We assume that in that case, $\mathcal{F}(h,R)$ is connected. In what follows, we also suppose that the center of mass of $\mathcal{S}$ is located at the origin so that h is the center of mass of $\mathcal{S}(h,R)$.

[*]Laboratoire LMA, UMR CNRS 5142, Université de Pau et des Pays de l'Adour, 64013 Pau Cedex, France (mehdi.badra@univ-pau.fr).

[†]Institut Élie Cartan, UMR 7502, INRIA, Université de Lorraine, CNRS, BP239, 54506 Vandœuvre-lès-Nancy Cedex, France

[‡]Team-Project CORIDA. INRIA Nancy - Grand Est, 615, rue du Jardin Botanique. 54600 Villers-lès-Nancy. France (takeo.takahashi@inria.fr).

In the bidimensional case ($d = 2$), $R = R_\theta$, with $\theta \in \mathbb{R}$, and with the notation

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

In particular, for $d = 2$, we will rather write $\mathcal{F}(h, \theta)$ instead of $\mathcal{F}(h, R_\theta)$.

If the rigid body follows the trajectory $t \mapsto (h(t), R(t))$, we can introduce its angular velocity. For $d = 2$, it is given by $r = \theta'$ whereas for $d = 3$, it is given through the formula

$$R'(t) = \mathbb{S}(r(t))R(t),$$

where

$$\mathbb{S}(r) = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} \quad (r \in \mathbb{R}^3).$$

Finally, the linear velocity of the rigid body is denoted by $V \stackrel{\text{def}}{=} h'$.

We present now the equations of motion of the fluid–structure system. These equations are valid as long as $\mathcal{S}(h(t), R(t)) \subset \Omega$.

In $\mathcal{F}(h(t), R(t))$ the fluid is described by a velocity vector field $v(t, x)$ and a pressure function $p(t, x)$ satisfying the incompressible Navier-Stokes equations:

$$\left.\begin{array}{rcl} \partial_t v + (v \cdot \nabla)v - \nu\Delta v + \nabla p &=& f^S \\ \operatorname{div} v &=& 0 \end{array}\right\} \quad \text{in } \mathcal{F}(h(t), R(t)) \quad t > 0, \tag{1.1}$$

and we suppose that it is subjected to a control u through the boundary condition:

$$v(t, x) = b^S + u(t, x), \quad t > 0, \ x \in \partial\Omega. \tag{1.2}$$

Moreover, we suppose that velocities coincide at the interface fluid-solid:

$$v(t, x) = V(t) + r(t) \times (x - h(t)), \quad x \in \partial\mathcal{S}(h(t), R(t)). \tag{1.3}$$

To obtain the equations of the positions we apply the Newton's laws by distinguishing the external forces/torques and the force/torque coming from the fluid that are expressed through the Cauchy stress tensor:

$$MV' = -\int_{\partial\mathcal{S}(h(t),R(t))} \mathbb{T}(v, p)n \, d\Gamma + f_M^S, \quad t > 0, \tag{1.4}$$

$$(Ir)' = -\int_{\partial\mathcal{S}(h(t),R(t))} (x - h) \times \mathbb{T}(v, p)n \, d\Gamma + f_I^S, \quad t > 0. \tag{1.5}$$

In above settings, we use the notation

$$\mathbb{T}(v, p) \stackrel{\text{def}}{=} 2\nu D(v) - p \operatorname{Id}, \quad \text{with} \quad D(u) \stackrel{\text{def}}{=} \frac{1}{2}\left((\nabla u) + {}^t(\nabla u)\right), \tag{1.6}$$

and, to simplify the study, we assume that the density $\rho_\mathcal{S}$ of the rigid body is a positive constant so that the quantities M and I are defined by

$$M \stackrel{\text{def}}{=} \rho_\mathcal{S} \, |\mathcal{S}|$$

and

$$I(t) = I(h(t), R(t)) \stackrel{\text{def}}{=} \rho_\mathcal{S} \int_{\mathcal{S}(h(t),R(t))} \left(|x - h(t)|^2 - (x - h(t)) \otimes (x - h(t))\right) dx. \tag{1.7}$$

Finally, we assume the following initial conditions for the positions:

$$h(0) = h^0, \quad R(0) = R^0, \quad V(0) = V^0, \quad r(0) = r^0, \quad v(0, x) = v^0(x) \quad x \in \mathcal{F}(h^0, R^0). \tag{1.8}$$

In the bidimensional case, the equations are written differently since r is a scalar function. We write $x^\perp \stackrel{\text{def}}{=} R_{\pi/2}x = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$, we replace (1.3) by

$$v(t, x) = V(t) + r(t)(x - h(t))^\perp, \quad x \in \partial\mathcal{S}(h(t), \theta(t)), \tag{1.9}$$

we replace (1.5) by

$$Ir' = -\int_{\partial\mathcal{S}(h(t),\theta(t))} (x-h)^\perp \cdot \mathbb{T}(v,p)n \, d\Gamma + f_I^S, \quad t>0 \tag{1.10}$$

and we replace (1.8) by

$$h(0)=h^0, \quad \theta(0)=\theta^0, \quad V(0)=V^0, \quad r(0)=r^0, \quad v(0,x)=v^0(x) \quad x\in\mathcal{F}(h^0,\theta^0). \tag{1.11}$$

Note that for $d=2$, I is independent of h,θ and is given by

$$I = \rho_{\mathcal{S}} \int_{\mathcal{S}} |y|^2 \, dy.$$

Let us emphasize that the unknowns of system (1.1)–(1.11) are the fluid velocity and pressure, but also the domains $\mathcal{F}(h(t),R(t))$ and $\mathcal{S}(h(t),R(t))$ which may evolve due to the dynamics induced by equations (1.4) and (1.5). This implies in particular that in (1.1) the equations are satisfied in an open non-cylindrical set in $\mathbb{R}^{d+1}$ depending on the unknown trajectories $t\mapsto(h(t),R(t))$. This is one of the main difficulties of this problem as explained in more details below.

In (1.1), (1.4), (1.5), the quantities $f^S=f^S(x)$, f_M^S, f_I^S and the boundary condition b^S are given and are independent of time. They correspond to a stationary solution (v^S,p^S,h^S,R^S) i.e. to a solution of the following problem:

$$\left\{\begin{array}{l}
(v^S\cdot\nabla)v^S - \nu\Delta v^S + \nabla p^S = f^S \quad \text{in } \mathcal{F}(h^S,R^S),\\
\operatorname{div} v^S = 0 \quad \text{in } \mathcal{F}(h^S,R^S),\\
v^S(x)=0, \quad x\in\partial\mathcal{S}(h^S,R^S),\\
v^S(x)=b^S, \quad x\in\partial\Omega,\\
-\displaystyle\int_{\partial\mathcal{S}(h^S,R^S)} \mathbb{T}(v^S,p^S)n\, d\Gamma + f_M^S = 0,\\
-\displaystyle\int_{\partial\mathcal{S}(h^S,R^S)} (x-h^S)\times\mathbb{T}(v^S,p^S)n\, d\Gamma + f_I^S = 0.
\end{array}\right. \tag{1.12}$$

In the bidimensional case, we replace the last above equality by

$$-\int_{\partial\mathcal{S}(h^S,\theta^S)} (x-h^S)^\perp\cdot\mathbb{T}(v^S,p^S)n\, d\Gamma + f_I^S = 0.$$

In the above equations, v^S is the velocity of the fluid and, since we consider a stationary solution, we assume it is independent of time and we have imposed that the velocity of the solid is 0, so that the solid remains fixed at position (h^S,R^S). Because of this condition, the system is overdetermined and for general f^S, f_M^S, f_I^S and b^S, there are no solutions to (1.12). Nevertheless, one can construct families of (1.12) by considering a fixed (h^S,R^S) and a solution (v^S,p^S) of the stationary Navier–Stokes system corresponding to the first four equations of (1.12). Then, it is sufficient to define

$$f_M^S \overset{\text{def}}{=} \int_{\partial\mathcal{S}(h^S,R^S)} \mathbb{T}(v^S,p^S)n\, d\Gamma, \quad f_I^S \overset{\text{def}}{=} \int_{\partial\mathcal{S}(h^S,R^S)} (x-h^S)\times\mathbb{T}(v^S,p^S)n\, d\Gamma$$

to obtain a solution of (1.12). These may correspond to inner forces/torques of the object or to exterior forces/torques prescribed to withstand a given steady flow.

Here we aim at studying the boundary feedback stabilizability of v,h,R [resp. θ] around a stationary state v^S,h^S,R^S [resp. θ^S]. More precisely, for an initial data (v^0,V^0,r^0,h^0,R^0) [resp. θ^0] close to the steady state $(v^S,0,0,h^S,R^S)$ [resp. θ^S] at time $t=0$ we aim at finding a feedback control u in (1.2) such that the resulting solution $(v(t),V(t),r(t),h(t),R(t))$ [resp. $\theta(t)$] goes to $(v^S,0,0,h^S,R^S)$ [resp. θ^S] as $t\to+\infty$ with a prescribed exponential rate of decrease. To achieve this goal, we are going to use the general strategy described in [5, 7] which relies on the stabilizability of the system obtained by linearizing around the stationary solution.

However, a difficulty to stabilize the system (1.1), (1.2), (1.3), (1.4), (1.5) and that is not present in the classical Navier–Stokes system comes from the fact that the fluid system is written in a moving domain $\mathcal{F}(h(t), R(t))$. Moreover this domain can be different from the domain $\mathcal{F}(h^S, R^S)$ of the stationary solution. This problem is classical in the study of fluid–structure system, and several techniques were proposed to prove the existence of weak solutions or of strong solutions: [12], [14], [17], [22], [20], [33], [34], etc. In the case of strong solutions, a method quite natural consists in using a change of variables in order to rewrite the system in a cylindrical domain. This was used for instance in [20], in [34], but also in some controllability problems related to our work: [10], [23], [9]. Note also that this strategy was applied in the 1D case for a simplified fluid–particle system in the case of controllability ([15], [26]) or in the case of stabilization by the authors ([6]).

Stabilization problems and controllability problems are strongly connected and for instance, for parabolic systems one can prove some relations of equivalence between stabilizability and approximate controllability (see, for instance, [7]). Note however that the methods used to prove null-controllability in the above references are quite different to the method considered here to prove local stabilizability. Moreover, it should be underlined that stabilization problems are closer to applications since the idea is to construct feedback operators (for instance by solving Riccati equations) in order to stabilize the system. Indeed, since a feedback control is constructed regardless to the initial datum, it is more robust to initial fluctuations or to inaccuracies of the model.

Here we follow the approach based on a change of variables to stabilize the fluid-rigid body system in dimension 2 ($d = 2$) or in dimension 3 ($d = 3$). It is important to remark that in the above quoted papers using this method, the initial conditions are in H^1 for the fluid velocity. But, as it has been explained in [4] (see also [3, 2]), for the construction of a stabilizing feedback control this regularity for the initial condition may cause some difficulty due to the initial compatibility condition with the feedback control u. In the literature, the Navier–Stokes system is stabilized in the 2D case for initial data in L^2 (or at most $H^{\frac{1}{2}-\epsilon}$) with classical feedback operators; in the 3D case, the initial data are in H^1 (or at least $H^{\frac{1}{2}+\epsilon}$) and several techniques were introduced to overcome the problem of the initial compatibility conditions: [27], [2], [5]. For instance in [5], the solution consists in taking a control u satisfying an evolution equation with another control feedback. We are thus reduced to stabilize a system coupling the fluid velocity and the feedback control u.

In this paper, we aim at considering the stabilization of the system (1.1), (1.2), (1.3), (1.4), (1.5) for $d = 3$ by using the same approach as in [5], i.e. with dynamical controls u. For $d = 2$, we consider the analogous system but with an initial fluid velocity in L^2. In order to do this and to use a method based on a change of variables, the crucial point is to remark that tp and $t\partial_t v$ are time integrable even if p and $\partial_t v$ are not. Then under the assumption that the initial perturbations of the positions $h^0 - h^S$ and $\theta^0 - \theta^S$ are zero the nonlinear terms involving the pressure and the velocity time derivative can be suitably estimated. Of course, we must underline that such initial restrictions on the positions are proper to feedback control and disappear if we consider a dynamical control.

Let us emphasize that, unlike in previous works on the subject, we consider here a stationary pair (v^S, p^S) that is not necessarily equal to zero. Indeed, there are numerous papers dealing with the controllability of fluid-solid interaction systems such as [15, 23, 10, 30, 31, 9, 26] and if we except the first recent 1D-result [6] of the authors, up to our knowledge there is no controllability or stabilizability result to non zero trajectories. The same remark holds for more complex systems modelizing deformable solids, see for instance [28, 25]. The main consequence of assuming non zero v^S, p^S is that the linearized system contains some additional position terms in the velocity equation. Then to prove the observability of the adjoint fluid-solid system we have to take into account some global velocity terms in the positions equations of the adjoint system. In fact, these global terms are easily treated by using the infinite dimensional Hautus–Fattorini test for stabilizability introduced in [5, 7].

In order to state our main results, we first need to introduce some notations. In the following, we consider the classical Lebesgue and Sobolev spaces L^α, H^k, and C_b stands for the continuous and bounded maps. Moreover, we use the bold notation for the spaces of vector fields: $\mathbf{L}^\alpha = (L^\alpha)^d$, $\mathbf{H}^k = (H^k)^d$ etc. We also use functional spaces of type $L^2(0,\infty; \mathbf{H}^1(\mathcal{F}(h(t), \theta(t))))$. This is a notation that can be precise: if $(h, \theta) \in H^1(0,\infty; \mathbb{R}^2 \times \mathbb{R})$ one can construct a change of variables $X \in H^1(0,\infty; \mathbb{R}^2)$ so that $X(t, \mathcal{F}(h^S, \theta^S)) = \mathcal{F}(h(t), \theta(t)))$ for all t. We say that $v \in L^2(0,\infty; \mathbf{H}^1(\mathcal{F}(h(t), \theta(t))))$ if $v \circ X \in L^2(0,\infty; \mathbf{H}^1(\mathcal{F}(h^S, \theta^S)))$. It can be seen that this definition is independent of the choice of X. Other spaces of functions defined on a non cylindrical domain of $\mathbb{R}^{d+1}$ are defined similarly: $C_b([0,\infty); \mathbf{L}^2(\mathcal{F}(h(t), \theta(t))))$, $H^1(0,\infty; \mathbf{L}^2(\mathcal{F}(h(t), \theta(t))))$, etc.

Let us now give a precise definition of our solutions for the system (1.1)-(1.8). This definition is only needed in the 2D case.

Definition 1. *Assume $d = 2$. A solution (v, p, V, r, h, θ) of* (1.1)*,* (1.2)*,* (1.9)*,* (1.4)*,* (1.10) *is a 6-tuple such that*

$$(h, \theta) \in H^1(0, \infty; \mathbb{R}^2 \times \mathbb{R}), \tag{1.13}$$

$$v \in L^2(0, \infty; \mathbf{H}^1(\mathcal{F}(h(t), \theta(t)))) \cap C_b([0, \infty); \mathbf{L}^2(\mathcal{F}(h(t), \theta(t))))$$

$$tv \in L^2(0, \infty; \mathbf{H}^2(\mathcal{F}(h(t), \theta(t)))) \cap C_b([0, \infty); \mathbf{H}^1(\mathcal{F}(h(t), \theta(t))) \cap H^1(0, \infty; \mathbf{L}^2(\mathcal{F}(h(t), \theta(t)))),$$

$$t\nabla p \in L^2(0, \infty; \mathbf{L}^2(\mathcal{F}(h(t), \theta(t))))$$

and (1.1)*,* (1.2)*,* (1.9)*,* (1.4)*,* (1.10) *are satisfies almost everywhere or in the trace sense.*

As already explained above, in the 2D case, due to difficulties linked to the time regularity of the pressure and of the time derivative of the velocity, we assume

$$h^0 = h^S \quad \text{and} \quad \theta^0 = \theta^S. \tag{1.14}$$

Then the initial domains $\mathcal{F}(h^0, \theta^0)$, $\mathcal{S}(h^0, \theta^0)$ coincide with the stationary ones $\mathcal{F}(h^S, \theta^S)$, $\mathcal{S}(h^S, \theta^S)$. It means that the initial positions are not allowed to be perturbed at time $t = 0$ but only the velocities of the fluid and of the object. As a consequence, v^0 and v^S can easily be compared since they are defined in the same domain $\mathcal{F}(h^S, \theta^S)$. However, it is not the case for $v(t)$ and v^S which are respectively defined in $\mathcal{F}(h(t), \theta(t))$ and in $\mathcal{F}(h^S, \theta^S)$. To compare them, one can use a change of variables. It is also possible to extend the fluid velocity by the rigid body velocity in the solid domain:

$$v(t, x) \stackrel{\text{def}}{=} V(t) + r(t)(x - h(t))^{\perp} \quad \text{in} \quad \mathcal{S}(h(t), \theta(t)),$$

$$v^S(x) \stackrel{\text{def}}{=} 0 \quad \text{in} \quad \mathcal{S}(h^S, \theta^S).$$

In that case, we can also compare v and v^S, but it is important to notice that the global velocity fields may be non-regular even if v or v^S are smooth in the fluid domains and in the structure domains.

Let us now introduce some notation corresponding to the stabilizability. We fix a desired rate of decrease $\sigma > 0$ and we search for a family of shape functions $\{v_j \in \mathbf{L}^2(\partial\Omega) \; ; \; j = 1, \ldots, N_\sigma\}$ and of a family of kernels $\{(\varphi_j, \xi_j, \zeta_j, a_j, b_j) \in \mathbf{L}^2(\mathcal{F}(h^S, \theta^S)) \times \mathbb{R}^6 \; ; \; j = 1, \ldots, N_\sigma\}$ such that the solution of (1.1), (1.9), (1.4), (1.10), (1.11) and of

$$\begin{aligned} v(t, x) = b^S + \sum_{j=1}^{N_\sigma} \bigg(\int_{\mathcal{F}(h^S, \theta^S)} & \operatorname{Cof}(\nabla X(t, y))^* v(t, X(t, y)) - v^S(y)) \cdot \varphi_j(y) dy + M R_{\theta^S - \theta(t)} V(t) \cdot \xi_j \\ & + I r(t) \zeta_j + (h(t) - h^S) \cdot a_j + (\theta(t) - \theta^S) b_j \bigg) v_j(x), \quad t > 0, \; x \in \partial\Omega, \end{aligned} \tag{1.15}$$

satisfies

$$\|v(t) - v^S\|_{\mathbf{L}^2(\Omega)} + |h(t) - h^S| + |\theta(t) - \theta^S| \leqslant C e^{-\sigma t} \|v^0 - v^S\|_{\mathbf{L}^2(\Omega)}. \tag{1.16}$$

We are now in position to state our first main result. The following theorem is a direct consequence of Theorem 26 of Subsection 3.3 below.

Theorem 2. *Assume that $d = 2$, that $f^S \in \mathbf{W}^{2,\infty}(\mathcal{F}(h^S, \theta^S))$ and that (f^S, f^S_M, f^S_I, b^S) is associated with a stationary solution $(v^S, p^S, 0, 0, h^S, \theta^S)$ such that $(v^S, p^S) \in \mathbf{W}^{2,\infty}(\mathcal{F}(h^S, \theta^S)) \times W^{1,\infty}(\mathcal{F}(h^S, \theta^S))$. For all $\sigma > 0$, there exist $N_\sigma \in \mathbb{N}$, $c_0, C > 0$, $v_j \in \mathbf{L}^2(\partial\Omega)$ and $(\varphi_j, \xi_j, \zeta_j, a_j, b_j) \in \mathbf{L}^2(\mathcal{F}(h^S, \theta^S)) \times \mathbb{R}^6$, $j = 1, \ldots, N_\sigma$ such that if* (1.14) *holds and $(v^0, V^0, r^0) \in \mathbf{L}^2(\mathcal{F}(h^S, \theta^S)) \times \mathbb{R}^3$ satisfies*

$$\|v^0 - v^S\|_{\mathbf{L}^2(\Omega)} \leqslant c_0,$$

then there exists a solution (v, p, V, r, h, θ) of (1.1)*,* (1.9)*,* (1.4)*,* (1.10)*,* (1.11)*,* (1.15) *(in the sense of Definition 1) satisfying* (1.16)*.*

Remark 3. *Let us remark that in the above theorem, the feedback control u is located on the whole boundary $\partial\Omega$. The same result holds for a control located on a nonempty open part Γ_c of $\partial\Omega$. The changes in the proof of Theorem 2 are only technical and rely on the use of a suitable cut-off function $m \in C^2(\partial\Omega)$ with support Γ_c. The point is to replace* (1.2) *by*

$$v(t,x) = b^S + \mathcal{M}(u)(t,x), \quad t > 0,\ x \in \partial\Omega, \tag{1.17}$$

where $\mathcal{M}$ is the localization operator introduced in [32] and defined as follows:

$$\mathcal{M}(u)(x) \stackrel{def}{=} m(x)u(x) - \frac{m(x)}{\int_{\partial\Omega} m d\Gamma}\left(\int_{\partial\Omega} mu \cdot n d\Gamma\right) n(x).$$

In (1.17) *the control $\mathcal{M}(u)$ is supported in Γ_c and has a normal component of zero mean to be compatible with the zero divergence condition in $\mathcal{F}$. The proof of Theorem 2 for a control of the form* (1.17) *is the same. Simply note that the unique continuation result yielding the stabilizability of the linearized system is obtained from an overdetermined condition (see* (3.34)*) which now holds only on Γ_c.*

Remark 4. *The regularity assumption $\partial\Omega$ of class $C^{2,1}$ is needed because we consider a control which is not necessarily tangential. Indeed, the semigroup formulation associated to a non tangential Dirichlet condition involves a quasi-stationary equation (see* (3.12)*) for which the H^2-regularity of the solution relies on the $C^{2,1}$ regularity of $\partial\Omega$ (see Proposition 15 relying on regularity results for Neumann problem* (3.16)*).*

Remark 5. *Note that Theorem 2 does not guarantee the uniqueness of the controlled solution. To be precise, since the proof relies on a Banach fixed point argument the uniqueness of the solution is true within a class of stable solutions sufficiently close to the stationary state (see Remark 29 below). However, outside a neighborhood of the target state we do not know if there exist other trajectories subject to the feedback law* (1.15)*, stable or even unstable. In fact, the uniqueness of such a controlled weak solution is not an easy issue even under the hypothesis of small initial data and this must be the object of further investigations. In the uncontrolled case, the uniqueness of weak solutions has been proved very recently in [19]. Their result can not be applied in our case since they consider an homogeneous Dirichlet boundary condition that has to be replaced here by a boundary condition involving a feedback boundary control. Nevertheless, the method and the ideas of their proof might be adapted here to obtain the uniqueness result.*

Theorem 2 concerns only the 2D case and the "weak" solutions (initial data in L^2). In our proof, we are led to impose condition (1.14). For strong solutions in the 2D case (initial data in H^1), the initial trace and the initial value of the control must coincide. This means that, at initial time, a compatibility condition depending on the feedback law must be imposed, and therefore, it is not possible to construct a strong solution for a relevant class of initial data: we are led to adopt another strategy such as the use of "dynamical controllers". In that case, the condition (1.14) is no more necessary in the proof. In the 3D case, our method can not be used to consider "weak" solutions, even for the classical Navier–Stokes system (without any rigid body). Consequently, we also work with strong solutions with "dynamical controllers" and again condition (1.14) is not imposed. To be more precise, we state here the corresponding result in 3D (a similar 2D result can be given)

The approach is already developed and studied in [2] and [5]; hence, the proof of the result below is only sketched (see Section 5) since it can be obtained with the same estimates as in the 2D case, and following the functional framework proposed for instance in [5] (see also [7]) to take into account the dynamical controls.

The main idea consists in assuming that the control u in (1.2) can be written as

$$u(t,x) = \sum_{j=1}^{N_\sigma} u_j v_j, \tag{1.18}$$

with $\overline{u} \stackrel{\text{def}}{=} (u_j)_{j\in\{1,\ldots,N_\sigma\}}$ satisfying

$$\begin{aligned} \overline{u}' = \Lambda\overline{u} + \sum_{j=1}^{N_\sigma}\Bigg(\int_{\mathcal{F}(h^S,\theta^S)} &\operatorname{Cof}(\nabla X(t,y))^* v(t, X(t,y)) - v^S(y)) \cdot \varphi_j(y)dy + MR^S R(t)^* V(t)\cdot \xi_j \\ &+ R^S R(t)^* Ir(t)\cdot\zeta_j + (h(t)-h^S)\cdot a_j + (R(t)-R^S) : b_j\Bigg), \quad t>0, \end{aligned} \tag{1.19}$$

$$\overline{u}(0) = 0,$$

for a suitable family $\{(\varphi_j, \xi_j, \zeta_j, a_j, b_j) \in \mathbf{L}^2(\mathcal{F}(h^S, \theta^S)) \times \mathbb{R}^9 \times \mathbb{R}^{3\times 3} \, ; \, j = 1, \dots, N_\sigma\}$ and matrix Λ of size $N_\sigma \times N_\sigma$. In above setting, $\{e_j \, ; \, j = 1, \dots, N_\sigma\}$ denotes a basis of $\mathbb{R}^{N_\sigma}$.

Theorem 6. *Assume that $d = 3$, that $f^S \in \mathbf{W}^{2,\infty}(\mathcal{F}(h^S, R^S))$ and that (f^S, f^S_M, f^S_I, b^S) is associated with a stationary solution $(v^S, p^S, 0, 0, h^S, R^S)$ such that $(v^S, p^S) \in \mathbf{W}^{2,\infty}(\mathcal{F}(h^S, R^S)) \times W^{1,\infty}(\mathcal{F}(h^S, R^S))$. For all $\sigma > 0$, there exist $N_\sigma \in \mathbb{N}$, $c_0, C > 0$, $v_j \in \mathbf{L}^2(\partial\Omega)$ and $(\varphi_j, \xi_j, \zeta_j, a_j, b_j) \in \mathbf{L}^2(\mathcal{F}(h^S, \theta^S)) \times \mathbb{R}^9 \times \mathbb{R}^{3\times 3}$, $j = 1, \dots, N_\sigma$, and $\Lambda \in \mathbb{R}^{N_\sigma \times N_\sigma}$ such that if $(v^0, V^0, r^0, h^0, R^0) \in \mathbf{H}^1(\mathcal{F}(h^0, R^0)) \times \mathbb{R}^9 \times SO(3)$ satisfies*

$$\operatorname{div} v^0 = 0 \quad \text{in } \mathcal{F}(h^0, R^0), \tag{1.20}$$

$$v^0 = V^0 + r^0 \times (x - h^0) \quad \text{on } \partial\mathcal{S}(h^0, R^0), \tag{1.21}$$

$$v^0 = b^S \quad \text{on } \partial\Omega, \tag{1.22}$$

and

$$\|v^0 - v^S\|_{\mathbf{H}^1(\Omega)} + |h^0 - h^S| + |R^0 - R^S| \leqslant c_0,$$

then there exists a strong solution (v, p, V, r, h, R, u)

$$v \in L^2(0, \infty; \mathbf{H}^2(\mathcal{F}(h(t), R(t)))) \cap C_b([0, \infty); \mathbf{H}^1(\mathcal{F}(h(t), R(t))) \cap H^1(0, \infty; \mathbf{L}^2(\mathcal{F}(h(t), R(t)))),$$
$$\nabla p \in L^2(0, \infty; \mathbf{L}^2(\mathcal{F}(h(t), R(t)))),$$
$$(V, r, h, R, u) \in H^1(0, \infty; \mathbb{R}^9 \times SO(3) \times \mathbb{R}^{N_\sigma})$$

of (1.1), (1.2), (1.3), (1.4), (1.5), (1.8), (1.18), (1.19) *such that*

$$\|v(t) - v^S\|_{\mathbf{H}^1(\Omega)} + |h(t) - h^S| + |R(t) - R^S| + |\overline{u}(t)| \leqslant Ce^{-\sigma t} \left(\|v^0 - v^S\|_{\mathbf{H}^1(\Omega)} + |h^0 - h^S| + |R^0 - R^S| \right).$$

Remark 7. *In contrast to the 2D-weak solutions, it is not difficult to prove that 3D-strong solutions given in Theorem 6 are unique within the class of solutions (v, p, V, r, h, R, u) in*

$$v \in L^2(0, T; \mathbf{H}^2(\mathcal{F}(h(t), R(t)))) \cap H^1(0, T; \mathbf{L}^2(\mathcal{F}(h(t), R(t)))),$$
$$\nabla p \in L^2(0, T; \mathbf{L}^2(\mathcal{F}(h(t), R(t)))),$$
$$(V, r, h, R, u) \in H^1(0, T; \mathbb{R}^9 \times SO(3) \times \mathbb{R}^{N_\sigma}),$$

for all $T > 0$. Indeed, the idea stands in using a change of variables similar to the one introduced in Section 5 to write both solutions in the same cylindrical domain and to use estimates analogous to the one obtained in Section 4 to compare the solutions. This can be easily achieved because the pressure function associated to strong solutions is not singular at $t = 0$. The proof is similar to the proof for the existence of the fixed point and we thus skip it here.

The outline of the paper is as follows. In Section 2, we introduce the change of variables and we rewrite the system in a fixed domain. Section 3 corresponds to the feedback stabilization: in Subsection 3.1 we introduce a semigroup formulation of the system written in a fixed domain; in Subsection 3.2 we introduce the stabilizing feedback law and we state regularity results for the closed-loop nonhomogeneous linear system; in Subsection 3.3 we use a fixed point procedure to obtain a solution of the whole nonlinear system. Section 4 is devoted to several technical and postponed proofs. In Section 5 we give some details about the 3D case.

2 Change of variables

In this section, we rewrite our system in a cylindrical domain by using a change of variables. Let us consider a change of variables $X(t, \cdot) : \Omega \to \Omega$, such that $X(t, \mathcal{F}(h^S, \theta^S)) = \mathcal{F}(h(t), \theta(t))$. We denote by $Y(t, \cdot)$ the inverse of $X(t, \cdot)$. Up to a rotation and a translation on $\mathcal{S}$, we can assume that

$$h^S = 0, \quad \theta^S = 0, \tag{2.1}$$

so that $\mathcal{S}(h^S, \theta^S) = \mathcal{S}$. In that case, we also write $\mathcal{F} = \mathcal{F}(h^S, \theta^S)$ and we have $X(t, \mathcal{F}) = \mathcal{F}(h(t), \theta(t))$. We impose moreover that $X(t, y) = y$ for y in a neighborhood of $\partial\Omega$ and that $X(t, y) = R_{\theta(t)} y + h(t)$ in a neighborhood of $\mathcal{S}$.

Several constructions are possible to obtain such a change of variables. Here, we consider

$$X(t,y) \stackrel{\text{def}}{=} y + \eta(y)\left[h(t) + (R_{\theta(t)} - I_2)y\right] \tag{2.2}$$

where η is a smooth function which is equal to 1 in $\mathcal{S}^\varepsilon$ and 0 on $\mathbb{R}^2 \backslash \mathcal{S}^{2\varepsilon}$. Here we have denoted by $\mathcal{S}^\varepsilon$ the domain

$$\mathcal{S}^\varepsilon \stackrel{\text{def}}{=} \left\{ y \in \mathbb{R}^2 \; ; \; \operatorname{dist}(y, \mathcal{S}) < \varepsilon \right\}$$

and we have chosen ε small enough so that $\overline{\mathcal{S}^{2\varepsilon}} \subset \Omega$.

The map X is a C^∞-diffeomorphism of $\overline{\Omega}$ onto itself if

$$\|\eta\|_{W^{1,\infty}(\Omega)} \left(|h(t)| + |\theta(t)|\right) < c, \tag{2.3}$$

for c small enough. It satisfies $X(t, \mathcal{F}) = \mathcal{F}(h(t), \theta(t))$ and the other hypotheses used above.

In what follows, we assume that

$$\forall t \geqslant 0, \quad |h(t)| + |\theta(t)| < C_\star, \tag{2.4}$$

with $C_\star$ small enough so that (2.3) holds.

Remark 8. *We do not use the change of variables considered in [34]: one of the differences is the change of variables defined by* (2.2) *is not with Jacobian determinant equal to 1 but it seems simpler to manipulate here.*

Remark 9. *If we do not assume that* (2.1) *holds, then we would have to change the above definition by*

$$X(t,y) \stackrel{\text{def}}{=} y + \eta(y)\left[h(t) + R_{\theta(t)-\theta^S}(y - h^S) - y\right]$$

that transform $\mathcal{S}(h^S, \theta^S) = h^S + R_{\theta^S}\mathcal{S}$ onto $\mathcal{S}(h(t), \theta(t)) = h(t) + R_{\theta(t)}\mathcal{S}$ and is the identity in a neighborhood of $\partial\Omega$. Let us also remark that without hypothesis (1.14)*, $X(0, \cdot) \neq \mathrm{Id}$, indeed*

$$X(0,y) = y + \eta(y)\left[h^0 + R_{\theta^0 - \theta^S}(y - h^S) - y\right].$$

With the change of variables introduced above, we introduce the following notation

$$\tilde{v}(t,y) \stackrel{\text{def}}{=} \operatorname{Cof}(\nabla X(t,y))^* v(t, X(t,y)), \quad \tilde{p}(t,y) \stackrel{\text{def}}{=} p(t, X(t,y)), \tag{2.5}$$

$$\ell(t) \stackrel{\text{def}}{=} R_{-\theta(t)} V(t), \quad \omega(t) \stackrel{\text{def}}{=} r(t) \tag{2.6}$$

where $\operatorname{Cof}(M)$ is the cofactor matrix of M, which satisfies in particular $M(\operatorname{Cof}(M))^* = (\operatorname{Cof}(M))^* M = \det(M)\,\mathrm{Id}$.

Following [24], [34] and [11], we can prove that (v, p, V, r, h, θ) satisfies (1.1), (1.2), (1.9), (1.4), (1.10), (1.11) if and only if $(\tilde{v}, \tilde{p}, \ell, \omega, h, \theta)$ satisfies the following system

$$[\mathbf{K}\partial_t \tilde{v}] - \nu[\mathbf{L}\tilde{v}] + [\mathbf{M}\tilde{v}] + [\mathbf{N}\tilde{v}] + [\mathbf{G}\tilde{p}] = \tilde{f}^S \quad \text{in } (0,+\infty) \times \mathcal{F}, \tag{2.7}$$

$$\operatorname{div} \tilde{v} = 0 \quad \text{in } (0,+\infty) \times \mathcal{F}, \tag{2.8}$$

$$\tilde{v} = \ell + \omega y^\perp \quad \text{on } (0,+\infty) \times \partial\mathcal{S}, \tag{2.9}$$

$$\tilde{v} = b^S + u \quad \text{on } (0,+\infty) \times \partial\Omega, \tag{2.10}$$

$$M[R_\theta \ell]' = -R_\theta \int_{\partial\mathcal{S}} \mathbb{T}(\tilde{v}, \tilde{p}) n \, d\Gamma + f_M^S, \quad t > 0, \tag{2.11}$$

$$I\omega'(t) = -\int_{\partial\mathcal{S}} y^\perp \cdot \mathbb{T}(\tilde{v}, \tilde{p}) n \, d\Gamma + f_I^S, \quad t > 0, \tag{2.12}$$

$$h' = R_\theta \ell, \quad t > 0, \tag{2.13}$$

$$\theta' = \omega, \quad t > 0, \tag{2.14}$$

$$h(0) = h^0, \quad \theta(0) = \theta^0, \quad \ell(0) = \ell^0, \quad \omega(0) = \omega^0, \quad \tilde{v}(0,x) = \tilde{v}^0(x) \quad x \in \mathcal{F}.$$

The precise result for this equivalence is stated in Proposition 10 below.

The function $\tilde{f}^S$ is given by

$$\tilde{f}^S(t,y) \stackrel{\text{def}}{=} f^S(X(t,y)),$$

and the initial conditions are given by

$$\ell^0 \stackrel{\text{def}}{=} R_{-\theta^0} V^0, \quad \omega^0 \stackrel{\text{def}}{=} r^0, \quad \tilde{v}^0(y) \stackrel{\text{def}}{=} \operatorname{Cof}(\nabla X(0,y))^* v^0(X(0,y)).$$

The operators $\mathbf{K}$, $\mathbf{L}$, $\mathbf{G}$, and $\mathbf{N}$ depend on h and θ and the operator $\mathbf{M}$ depends on h, θ, ℓ and ω through the change of variables X and its inverse Y. Their definitions are given through the following formulas

$$[\mathbf{K}\tilde{v}] \stackrel{\text{def}}{=} (\operatorname{Cof}(\nabla Y)^* \circ X)\tilde{v}, \tag{2.15}$$

$$[\mathbf{M}\tilde{v}] \stackrel{\text{def}}{=} \frac{\partial}{\partial t}\left[\operatorname{Cof}(\nabla Y)^* \circ X\right]\tilde{v} + (\operatorname{Cof}(\nabla Y)^* \circ X)(\nabla\tilde{v})(\partial_t Y) \circ X, \tag{2.16}$$

$$\begin{aligned}
[\mathbf{L}\tilde{v}]_i \stackrel{\text{def}}{=} & \sum_{j,k} \frac{\partial^2}{\partial x_j^2} \operatorname{Cof}(\nabla Y)_{ki}(X)\tilde{v}_k + 2\sum_{j,k,l} \frac{\partial}{\partial x_j} \operatorname{Cof}(\nabla Y)_{ki}(X) \frac{\partial \tilde{v}_k}{\partial y_l}\frac{\partial Y_l}{\partial x_j}(X) \\
& + \sum_{j,k,l,m} \operatorname{Cof}(\nabla Y)_{ki}(X) \frac{\partial^2 \tilde{v}_k}{\partial y_l \partial y_m}\frac{\partial Y_l}{\partial x_j}(X)\frac{\partial Y_m}{\partial x_j}(X) + \sum_{j,k,l} \operatorname{Cof}(\nabla Y)_{ki}(X)\frac{\partial \tilde{v}_k}{\partial y_l}\frac{\partial^2 Y_l}{\partial x_j^2}(X),
\end{aligned} \tag{2.17}$$

$$[\mathbf{N}\tilde{v}]_i \stackrel{\text{def}}{=} \sum_{j,k,r} \operatorname{Cof}(\nabla Y)_{kj}(X)\frac{\partial}{\partial x_j}\operatorname{Cof}(\nabla Y)_{ri}(X)\tilde{v}_k\tilde{v}_r + \sum_{k,r} \det((\nabla Y)(X))^2 \frac{\partial X_i}{\partial y_r}\tilde{v}_k\frac{\partial \tilde{v}_r}{\partial y_k}, \tag{2.18}$$

$$[\mathbf{G}\tilde{p}]_i \stackrel{\text{def}}{=} \sum_l \frac{\partial \tilde{p}}{\partial y_l}\frac{\partial Y_l}{\partial x_i}(X). \tag{2.19}$$

We underline that we should rather write $\mathbf{K}(h,\theta)$, $\mathbf{L}(h,\theta)$, $\mathbf{G}(h,\theta)$, $\mathbf{N}(h,\theta)$, $\mathbf{M}(h,\theta,\ell,\omega)$ instead of $\mathbf{K}$, $\mathbf{L}$, $\mathbf{G}$, $\mathbf{N}$, $\mathbf{M}$ but we omit this dependence to simplify the writing. The following proposition is the main tool to deduce system (2.7)–(2.14). We omit its proof since it was already done for instance in [11].

Proposition 10. *Assume* $\mathbf{K}$, $\mathbf{M}$, $\mathbf{L}$, $\mathbf{N}$ *and* $\mathbf{G}$ *are given by* (2.15), (2.16), (2.17), (2.18) *and* (2.19). *Suppose also that* (2.5) *holds. Then*

$$\begin{aligned}
\operatorname{div} v &= \det(\nabla Y)\ (\operatorname{div}\tilde{v}) \circ Y, \\
\partial_t v &= [\mathbf{K}\partial_t \tilde{v}] \circ Y + [\mathbf{M}\tilde{v}] \circ Y, \\
\Delta v &= [\mathbf{L}\tilde{v}] \circ Y, \\
[(v\cdot\nabla)v] &= [\mathbf{N}\tilde{v}] \circ Y, \\
\nabla p &= [\mathbf{G}\tilde{p}] \circ Y.
\end{aligned}$$

Remark 11. *Let us note that* $(v^S, p^S, 0, 0, h^S, \theta^S)$ *is also a stationary solution of the system obtained after the change of variables. Indeed, if we assume* (1.14) *to simplify, system* (2.7)–(2.14) *reduces to* (1.12) *since* $h(t) = h^S = 0$ *and* $\theta(t) = \theta^S = 0$ *and the definition of* $X(t,\cdot)$ *guarantees* $X(t,\cdot) = \operatorname{Id}$.

Writing $\tilde{v} = w + v^S$, $\tilde{p} = q + p^S$, we deduce from (2.7) that

$$\begin{aligned}
\partial_t w - \nu\Delta w - \nu[\mathbf{L}v^S] + [\mathbf{M}v^S] + [\mathbf{N}(w+v^S)] + \nabla q + [\mathbf{G}p^S] \\
= \tilde{f}^S + [(\operatorname{Id} - \mathbf{K})\partial_t w] + \nu[(\mathbf{L}-\Delta)w] - [\mathbf{M}w] + [(\nabla - \mathbf{G})q]
\end{aligned}$$

and thus by using (1.12), we deduce

$$\begin{aligned}
\partial_t w - \nu\Delta w - \nu[(\mathbf{L}-\Delta)v^S] + [\mathbf{M}v^S] + [\mathbf{N}(w+v^S)] - (v^S\cdot\nabla)v^S + \nabla q + [(\mathbf{G}-\nabla)p^S] \\
= (\tilde{f}^S - f^S) + [(\operatorname{Id} - \mathbf{K})\partial_t w] + \nu[(\mathbf{L}-\Delta)w] - [\mathbf{M}w] + [(\nabla - \mathbf{G})q].
\end{aligned} \tag{2.20}$$

In what follows, we precise the "principal linear part" of (2.20) that is important for our stabilization analysis.

Proposition 12. *Assume $f^S \in \mathbf{W}^{2,\infty}(\mathcal{F})$ and assume that $(v^S,p^S) \in \mathbf{W}^{2,\infty}(\mathcal{F}) \times W^{1,\infty}(\mathcal{F})$ satisfies* (1.12) *with $h^S = 0$ and $\theta^S = 0$. Then there exist γ_L, γ_G in $L^\infty(\mathcal{F};\mathcal{L}(\mathbb{R}^3,\mathbb{R}^2))$, γ_f in $W^{1,\infty}(\mathcal{F};\mathcal{L}(\mathbb{R}^3,\mathbb{R}^2))$, γ_M in $L^\infty(\mathcal{F};\mathcal{L}(\mathbb{R}^6,\mathbb{R}^2))$, ε_L, ε_G, ε_f in $L^\infty(\mathcal{F};C^\infty(\mathbb{R}^3;\mathbb{R}^2))$, ε_M in $L^\infty(\mathcal{F};C^\infty(\mathbb{R}^6;\mathbb{R}^2))$ and a constant $C>0$ such that for all $(\ell,\omega,h,\theta)\in\mathbb{R}^6$ satisfying* (2.4) *the following equalities hold*

$$\begin{aligned} -\nu[(\mathbf{L}-\Delta)v^S] &= \gamma_L(\cdot)(h,\theta)+\varepsilon_L(\cdot,h,\theta), &(2.21)\\ [\mathbf{M}v^S] &= \gamma_M(\cdot)(h,\theta,\ell,\omega)+\varepsilon_M(\cdot,h,\theta,\ell,\omega), &(2.22)\\ [(\mathbf{G}-\nabla)p^S] &= \gamma_G(\cdot)(h,\theta)+\varepsilon_G(\cdot,h,\theta), &(2.23)\\ \tilde{f}^S-f^S &= \gamma_f(\cdot)(h,\theta)+\varepsilon_f(\cdot,h,\theta), &(2.24)\end{aligned}$$

and the following estimates hold

$$\|\varepsilon_L(\cdot,h,\theta)\|_{\mathbf{L}^\infty(\mathcal{F})}+\|\varepsilon_G(\cdot,h,\theta)\|_{\mathbf{L}^\infty(\mathcal{F})}+\|\varepsilon_f(\cdot,h,\theta)\|_{\mathbf{L}^\infty(\mathcal{F})}\leqslant C(|h|^2+|\theta|^2), \tag{2.25}$$

$$\|\varepsilon_M(\cdot,h,\theta,\ell,\omega)\|_{\mathbf{L}^\infty(\mathcal{F})}\leqslant C(|h|^2+|\theta|^2+|\ell|^2+|\omega|^2). \tag{2.26}$$

In the above statement $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)$ stands for the space of linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$. We postpone the proof of the proposition to Subsection 4.1.

Proposition 13. *Assume that $(v^S,p^S)\in\mathbf{W}^{2,\infty}(\mathcal{F})\times W^{1,\infty}(\mathcal{F})$ satisfies* (1.12) *with $h^S=0$ and $\theta^S=0$. Then there exist $\gamma_N\in L^\infty(\mathcal{F};\mathcal{L}(\mathbb{R}^3,\mathbb{R}^2))$, $\varepsilon_N\in L^\infty(\mathcal{F};C^\infty(\mathbb{R}^3;\mathbb{R}^2))$ and $C>0$ such that for all $(h,\theta)\in\mathbb{R}^3$ satisfying* (2.4) *the following equality holds*

$$[\mathbf{N}(w+v^S)]=(v^S\cdot\nabla)v^S+\gamma_N(\cdot)(h,\theta)+\Big[(w\cdot\nabla)v^S+(v^S\cdot\nabla)w\Big]+[\mathbf{N}^S(w)]+\varepsilon_N(\cdot,h,\theta), \tag{2.27}$$

where

$$\begin{aligned}[\mathbf{N}^S(w)]_i \overset{def}{=} &[(w\cdot\nabla)w]_i+\sum_{j,k,r}\operatorname{Cof}(\nabla Y)_{kj}(X)\frac{\partial}{\partial x_j}\operatorname{Cof}(\nabla Y)_{ri}(X)\left(w_kw_r+w_kv_r^S+v_k^Sw_r\right)\\ &+\sum_{k,r}\left(\det((\nabla Y)(X))^2\frac{\partial X_i}{\partial y_r}-\delta_{ir}\right)\left(w_k\frac{\partial v_r^S}{\partial y_k}+v_k^S\frac{\partial w_r}{\partial y_k}+w_k\frac{\partial w_r}{\partial y_k}\right)\end{aligned} \tag{2.28}$$

and the following estimate holds

$$\|\varepsilon_N(\cdot,h,\theta)\|_{\mathbf{L}^\infty(\Omega)}\leqslant C(|h|^2+|\theta|^2).$$

We postpone the proof of the proposition to Subsection 4.1.

In what follows, we write

$$\Gamma(\cdot)(h,\theta,\ell,\omega)\overset{\text{def}}{=}\gamma_L(\cdot)(h,\theta)+\gamma_M(\cdot)(h,\theta,\ell,\omega)+\gamma_G(\cdot)(h,\theta)+\gamma_N(\cdot)(h,\theta)-\gamma_f(\cdot)(h,\theta),$$

where γ_L, γ_M, γ_G, γ_N and γ_f are the functions of Propositions 12 and of Proposition 13. Then Γ belongs to $L^\infty(\mathcal{F};\mathcal{L}(\mathbb{R}^6,\mathbb{R}^2))$ and there exist $\Gamma_i\in L^\infty(\mathcal{F};\mathbb{R}^2)$, $i=1,\dots,6$ such that

$$\Gamma(\cdot,h,\theta,\ell,\omega)=h_1\Gamma_1+h_2\Gamma_2+\theta\Gamma_3+\ell_1\Gamma_4+\ell_2\Gamma_5+\omega\Gamma_6. \tag{2.29}$$

Finally at the end, we obtain from (2.20)

$$\partial_t w-\nu\Delta w+\Gamma(h,\theta,\ell,\omega)+(w\cdot\nabla)v^S+(v^S\cdot\nabla)w+\nabla q=F(\mathbf{X},q)\quad\text{in }(0,+\infty)\times\mathcal{F}, \tag{2.30}$$

$$\operatorname{div} w=0\quad\text{in }(0,+\infty)\times\mathcal{F}, \tag{2.31}$$

$$w=\ell+\omega y^\perp\quad\text{on }(0,+\infty)\times\partial\mathcal{S}, \tag{2.32}$$

$$w=u\quad\text{on }(0,+\infty)\times\partial\Omega, \tag{2.33}$$

$$M\ell'=-\int_{\partial\mathcal{S}}\mathbb{T}(w,q)n\,d\Gamma-\theta(f_M^S)^\perp+\varepsilon_\ell(\mathbf{X}),\quad t>0, \tag{2.34}$$

$$I\omega'=-\int_{\partial\mathcal{S}}y^\perp\cdot\mathbb{T}(w,q)n\,d\Gamma,\quad t>0, \tag{2.35}$$

$$h'=\ell+\varepsilon_h(\mathbf{X}),\quad t>0, \tag{2.36}$$

$$\theta'=\omega,\quad t>0, \tag{2.37}$$

$$h(0)=h^0,\quad\theta(0)=\theta^0,\quad\ell(0)=\ell^0,\quad\omega(0)=\omega^0,\quad w(0,y)=w^0(y)\quad y\in\mathcal{F}. \tag{2.38}$$

In the above system, we have set

$$\mathbf{X} \stackrel{\text{def}}{=} \begin{bmatrix} w \\ \ell \\ \omega \\ h \\ \theta \end{bmatrix}$$

and

$$\varepsilon_h(\mathbf{X}) \stackrel{\text{def}}{=} (R_\theta - I_2)\ell, \quad \varepsilon_\ell(\mathbf{X}) \stackrel{\text{def}}{=} (R_{-\theta} - I_2 + \theta R_{\pi/2}) f_M^S - M\omega\ell^\perp, \tag{2.39}$$

$$\begin{aligned} F(\mathbf{X}, q) \stackrel{\text{def}}{=} & - [\mathbf{N}^S(w)] + [(\mathrm{Id} - \mathbf{K})\partial_t w] + \nu[(\mathbf{L} - \Delta)w] - [\mathbf{M}w] + [(\nabla - \mathbf{G})q] \\ & - (\varepsilon_L(h,\theta) + \varepsilon_M(h,\theta,\ell,\omega) + \varepsilon_N(h,\theta) + \varepsilon_G(h,\theta) - \varepsilon_f(h,\theta)), \end{aligned} \tag{2.40}$$

where ε_L, ε_M, ε_N, ε_G and ε_f are defined in Proposition 12 and Proposition 13.

3 Feedback Stabilizability of (2.30)–(2.38)

In this section, we show the feedback stabilizability of (2.30)–(2.38). We follow the same classical method as in [5]: first we show the feedback stabilizability of a linear system associated with (2.30)–(2.38). In order to do this, we introduce a functional setting using the semigroup theory. With this framework, we can apply the general Hautus-Fattorini criterion for stabilizability (see, for instance, [5, 7]). The last part of this section consists in showing that the feedback operator of the linear system permits to stabilize the nonlinear system (2.30)–(2.38). This last step is done by a fixed point argument.

Let us begin by giving some notation used in what follows. For a Hilbert space $\mathcal{X}$ and $0 < T \leqslant +\infty$, $L^2(0,T;\mathcal{X})$, $L^\infty(0,T;\mathcal{X})$ and $H^1(0,T;\mathcal{X})$ are usual vector-valued Lebesgue and Sobolev spaces and if $T = +\infty$ we use the shorter expressions $L^2(\mathcal{X}) \stackrel{\text{def}}{=} L^2(0,+\infty;\mathcal{X})$ and $H^1(\mathcal{X}) \stackrel{\text{def}}{=} H^1(0,+\infty;\mathcal{X})$. For two Hilbert spaces $\mathcal{X}$, $\mathcal{Y}$ we use the notation

$$W(\mathcal{X},\mathcal{Y}) \stackrel{\text{def}}{=} L^2(\mathcal{X}) \cap H^1(\mathcal{Y}).$$

If $\mathcal{Z}$ is a vector-valued function space of the time variable $t \geqslant 0$ we use the subscript σ in $\mathcal{Z}_\sigma$ to denote

$$\mathcal{Z}_\sigma \stackrel{\text{def}}{=} \left\{ \mathbf{X} \in \mathcal{Z} \ ; \ t \mapsto e^{\sigma t}\mathbf{X}(t) \in \mathcal{Z} \right\}. \tag{3.1}$$

For instance, for $\sigma > 0$

$$W_\sigma(\mathcal{X},\mathcal{Y}) \stackrel{\text{def}}{=} \left\{ \mathbf{X} \in L^2(\mathcal{X}) \cap H^1(\mathcal{Y}) \ ; \ t \mapsto e^{\sigma t}\mathbf{X}(t) \in W(\mathcal{X},\mathcal{Y}) \right\}.$$

We use the notation $\mathcal{L}(\mathcal{X},\mathcal{Y})$ for the bounded linear maps from $\mathcal{X}$ into $\mathcal{Y}$ and the notation $\mathcal{X} \hookrightarrow \mathcal{Y}$ for the continuous embedding of $\mathcal{X}$ into $\mathcal{Y}$.

We also introduce different spaces of free divergence functions: $\mathbf{V}_n^0(\mathcal{F})$ stands for the space of zero divergence functions f in $\mathbf{L}^2(\mathcal{F})$ which are tangential on $\partial\mathcal{F}$, i.e. $\operatorname{div} f = 0$ in $\mathcal{F}$ and $f \cdot n = 0$ on $\partial\mathcal{F}$; $\mathbf{V}_0^s(\mathcal{F})$ with $s > 1/2$ stands for the space of zero divergence functions f in $\mathbf{H}^s(\mathcal{F})$ such that $f = 0$ on $\partial\mathcal{F}$; we define the trace spaces:

$$\mathbf{V}^s(\partial\Omega) \quad \stackrel{\text{def}}{=} \quad \left\{ w \in \mathbf{H}^s(\partial\Omega) \ ; \ \langle w \cdot n, 1 \rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} = 0 \right\} \quad \text{for } s \geqslant -\frac{1}{2}.$$

3.1 A semigroup formulation

The present subsection is dedicated to the abstract formulation of the following linear nonhomogeneous problem:

$$\partial_t w - \nu\Delta w + \Gamma(h,\theta,\ell,\omega) + (w\cdot\nabla)v^S + (v^S\cdot\nabla)w + \nabla q = F \quad \text{in } (0,+\infty)\times\mathcal{F}, \tag{3.2}$$
$$\operatorname{div} w = 0 \quad \text{in } (0,+\infty)\times\mathcal{F}, \tag{3.3}$$
$$w = \ell + \omega y^\perp \quad \text{on } (0,+\infty)\times\partial\mathcal{S}, \tag{3.4}$$
$$w = u \quad \text{on } (0,+\infty)\times\partial\Omega, \tag{3.5}$$
$$M\ell' = -\int_{\partial\mathcal{S}} \mathbb{T}(w,q)n\ d\Gamma - \theta(f_M^S)^\perp + \ell_F, \quad t>0, \tag{3.6}$$
$$I\omega' = -\int_{\partial\mathcal{S}} y^\perp\cdot\mathbb{T}(w,q)n\ d\Gamma + \omega_F, \quad t>0, \tag{3.7}$$
$$h' = \ell + h_F, \quad t>0, \tag{3.8}$$
$$\theta' = \omega + \theta_F, \quad t>0, \tag{3.9}$$

with the initial conditions

$$h(0)=h^0, \quad \theta(0)=\theta^0, \quad \ell(0)=\ell^0, \quad \omega(0)=\omega^0, \quad w(0,y)=w^0(y) \quad y\in\mathcal{F}. \tag{3.10}$$

Here, F, ℓ_F, ω_F, h_F, θ_F are nonhomogeneous terms that replace the nonlinearities in (2.30)–(2.38) so that (3.2)–(3.9) is a linear system.

Let us show that the above system can be rewritten in the form

$$P\mathbf{X}' = AP\mathbf{X} + Bu + P\mathbf{F} \ \text{ in } [\mathcal{D}(A^*)]', \quad P\mathbf{X}(0) = P\mathbf{X}^0 \tag{3.11}$$
$$(I-P)\mathbf{X} = (I-P)D_{\mathcal{F}}u, \tag{3.12}$$

where $\mathbf{X} = [w,\ell,\omega,h,\theta]^*$, $\mathbf{F} = [F,\ell_F,\omega_F,h_F,\theta_F]^*$ and where A, P, B and $D_{\mathcal{F}}$ are adequate linear operators which are defined below. Formulation of type (3.11)-(3.12) is due to J.-P. Raymond in [27] for Stokes type system. Here, we aim to generalize such a formulation for the fluid-rigid body interaction system considered here.

For that, we consider the space $\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^6$ equipped with the scalar product:

$$\left\langle \begin{bmatrix} w^1\\ \ell^1\\ \omega^1\\ h^1\\ \theta^1 \end{bmatrix}, \begin{bmatrix} w^2\\ \ell^2\\ \omega^2\\ h^2\\ \theta^2 \end{bmatrix} \right\rangle = \int_{\mathcal{F}} w^1\cdot w^2\ dy + M\ell^1\cdot\ell^2 + I\omega^1\omega^2 + h^1\cdot h^2 + \theta^1\theta^2,$$

and we introduce the following subspaces:

$$\mathcal{H} \stackrel{\text{def}}{=} \left\{[w,\ell,\omega,h,\theta]^*\in\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^6\ ;\ \operatorname{div} w = 0 \text{ in } \mathcal{F},\ w\cdot n = (\ell+\omega y^\perp)\cdot n \text{ on } \partial\mathcal{S},\ w\cdot n = 0 \text{ on } \partial\Omega\right\},$$

$$\mathcal{V} \stackrel{\text{def}}{=} \left\{[w,\ell,\omega,h,\theta]^*\in\mathcal{H}\ ;\ w\in\mathbf{H}^1(\mathcal{F}),\ w=\ell+\omega y^\perp \text{ on } \partial\mathcal{S},\ w=0 \text{ on } \partial\Omega\right\}.$$

We have the following characterization of the orthogonal of $\mathcal{H}$ in $\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^6$.

Proposition 14. *The orthogonal of $\mathcal{H}$ in $\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^6$ is given by*

$$\mathcal{H}^\perp = \left\{\left[\nabla p, -M^{-1}\int_{\partial\mathcal{S}} pn\ d\Gamma, -I^{-1}\int_{\partial\mathcal{S}} pn\cdot y^\perp\ d\Gamma, 0, 0\right]^*\ ;\ p\in H^1(\mathcal{F})\right\} \tag{3.13}$$

Proof. Let suppose that $[w^1,\ell^1,\omega^1,a^1,\theta^1]^*\in\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^6$ satisfies for all $[w^2,\ell^2,\omega^2,a^2,\theta^2]^*\in\mathcal{H}$:

$$\int_{\mathcal{F}} w^1\cdot w^2\ dy + M\ell^1\cdot\ell^2 + I\omega^1\omega^2 + h^1\cdot h^2 + \theta^1\theta^2 = 0. \tag{3.14}$$

Then we have in particular that $\int_{\mathcal{F}} w^1 \cdot w^2 \, dy = 0$ for all $w^2 \in \mathbf{V}_0^1(\mathcal{F})$ and the De Rham's Lemma guarantees that $w^1 = \nabla p$ for some $p \in H^1(\mathcal{F})$, see [36, Chap. I, Prop. 1.1 and Rem 1.4]. Thus, by plugging $w^1 = \nabla p$ in (3.14) and integrating by parts, we obtain that

$$\int_{\partial \mathcal{S}} pn \cdot (\ell^2 + \omega^2 y^\perp) \, dy + M\ell^1 \cdot \ell^2 + I\omega^1\omega^2 + h^1 \cdot h^2 + \theta^1\theta^2 = 0,$$

is satisfied for all $(\ell^2, \omega^2, h^2, \theta^2) \in \mathbb{R}^6$, which gives the result. $\square$

Let us define P the orthogonal projection of $\mathbf{L}^2(\mathcal{F}) \times \mathbb{R}^6$ onto $\mathcal{H}$. It satisfies the following regularity properties.

Proposition 15. *The orthogonal projection operator $P : \mathbf{L}^2(\mathcal{F}) \times \mathbb{R}^6 \to \mathcal{H}$ satisfies:*

$$P \in \mathcal{L}(\mathbf{H}^s(\mathcal{F}) \times \mathbb{R}^6, \mathbf{H}^s(\mathcal{F}) \times \mathbb{R}^6), \quad s \in [0, 2]. \tag{3.15}$$

Proof. First, by using (3.13) we verify that for $[w, \ell, \omega, h, \theta]^* \in \mathbf{L}^2(\mathcal{F}) \times \mathbb{R}^6$:

$$P \begin{bmatrix} w \\ \ell \\ \omega \\ a \\ \theta \end{bmatrix} = \begin{bmatrix} w - \nabla p \\ \ell + M^{-1} \displaystyle\int_{\partial \mathcal{S}} pn \, d\Gamma \\ \omega + I^{-1} \displaystyle\int_{\partial \mathcal{S}} pn \cdot y^\perp \, d\Gamma \\ a \\ \theta \end{bmatrix}$$

where the pressure function $p \in H^1(\mathcal{F})$ is a solution to the Neumann problem

$$\begin{cases} \Delta p = \operatorname{div} w \text{ in } \mathcal{F}, \\ \dfrac{\partial p}{\partial n} = w \cdot n \text{ on } \partial\Omega, \\ \dfrac{\partial p}{\partial n} + \left(M^{-1} \displaystyle\int_{\partial \mathcal{S}} pn \, d\Gamma \right) \cdot n + I^{-1} \left(\displaystyle\int_{\partial \mathcal{S}} pn \cdot y^\perp \, d\Gamma \right) y^\perp \cdot n = w \cdot n - (\ell + \omega y^\perp) \cdot n \text{ on } \partial \mathcal{S}. \end{cases}$$

Note that the above Neumann problem is formal and must be interpreted in its weak form since $w \in \mathbf{L}^2(\mathcal{F})$. More precisely, the corresponding weak form of the above problem is

$$\begin{aligned} \int_{\mathcal{F}} \nabla p \cdot \nabla q \, dx + M^{-1} \left(\int_{\partial \mathcal{S}} pn \, d\Gamma \right) \cdot \left(\int_{\partial \mathcal{S}} qn \, d\Gamma \right) + I^{-1} \left(\int_{\partial \mathcal{S}} pn \cdot y^\perp \, d\Gamma \right) \left(\int_{\partial \mathcal{S}} qn \cdot y^\perp \, d\Gamma \right) \\ = \int_{\mathcal{F}} w \cdot \nabla q \, dx - \int_{\partial \mathcal{S}} (\ell + \omega y^\perp) \cdot nq \, d\Gamma. \end{aligned} \tag{3.16}$$

Using the Riesz theorem, we deduce that for all $w \in \mathbf{L}^2(\mathcal{F})$ and for $(\ell, \omega) \in \mathbb{R}^3$, there exists a unique solution $p \in H^1(\mathcal{F})/\mathbb{R}$. Then, since $\partial\mathcal{F}$ is of class $C^{2,1}$, regularity results for the Neumann problem gives (3.15). $\square$

Next, we define the linear operator $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ as follows: we set

$$\mathcal{D}(A) \overset{\text{def}}{=} \{ [w, \ell, \omega, h, \theta]^* \in \mathcal{V} \; ; \; w \in \mathbf{H}^2(\mathcal{F}) \}, \tag{3.17}$$

and for $[w, \ell, \omega, h, \theta]^* \in \mathcal{D}(A)$, we set

$$\widetilde{A} \begin{bmatrix} w \\ \ell \\ \omega \\ h \\ \theta \end{bmatrix} \overset{\text{def}}{=} \begin{bmatrix} \nu \Delta w - \Gamma(h, \theta, \ell, \omega) - (w \cdot \nabla) v^S - (v^S \cdot \nabla) w \\ -M^{-1} \displaystyle\int_{\partial \mathcal{S}} 2\nu D(w) n \, d\Gamma - M^{-1} \theta (f_M^S)^\perp \\ -I^{-1} \displaystyle\int_{\partial \mathcal{S}} y^\perp \cdot 2\nu D(w) n \, d\Gamma \\ \ell \\ \omega \end{bmatrix}$$

and

$$A \overset{\text{def}}{=} P\widetilde{A}. \tag{3.18}$$

We recall that $D(w)$ is the symmetric gradient of w defined in (1.6).

Proposition 16. *The operator A defined by* (3.17), (3.18) *is densely defined with compact resolvent and it is the infinitesimal generator of an analytic semigroup on $\mathcal{H}$.*

Proof. Since the two first statements are clear we only give a brief proof of the last claim. For that we introduce the following bilinear form on $\mathcal{V}$:

$$\mathbf{a}\left(\begin{bmatrix} w^1 \\ \ell^1 \\ \omega^1 \\ h^1 \\ \theta^1 \end{bmatrix}, \begin{bmatrix} w^2 \\ \ell^2 \\ \omega^2 \\ h^2 \\ \theta^2 \end{bmatrix}\right) \stackrel{\text{def}}{=} -\int_{\mathcal{F}} 2\nu D(w^1) : D(w^2) dy - \int_{\mathcal{F}} \Big[(w^1 \cdot \nabla) v^S + (v^S \cdot \nabla) w^1 + \Gamma(h^1, \theta^1, \ell^1, \omega^1)\Big] \cdot w^2 dy$$
$$- \theta^1 (f_M^S)^\perp \cdot \ell^2 + \ell^1 \cdot h^2 + \omega^1 \theta^2.$$

The bilinear form $-\mathbf{a}(\cdot, \cdot)$ is regularity accretive, i.e there exist $c_0 > 0$ and $\lambda_0 > 0$ such that

$$\lambda_0 \|\mathbf{X}\|_{\mathcal{H}}^2 - \mathbf{a}(\mathbf{X}, \mathbf{X}) \geqslant c_0 \|\mathbf{X}\|_{\mathcal{V}}^2. \tag{3.19}$$

Consequently, standard arguments (see [8, Thm. 2.12]) guarantee that the operator A_0 defined from $\mathbf{a}(\cdot, \cdot)$ is the infinitesimal generator of an analytic semigroup on $\mathcal{H}$. Thus, from regularity results for the Stokes problem with regular boundary data we deduce that $\mathcal{D}(A_0) = \mathcal{D}(A)$ (defined in (3.17)) and an integration by parts yields $A_0 = A$ given by (3.18) (see [35] for a similar proof). □

We have the following characterization of the adjoint of A.

Proposition 17. *The adjoint of the operator A is given by $\mathcal{D}(A^*) = \mathcal{D}(A)$ and*

$$A^* \begin{bmatrix} \varphi \\ \xi \\ \zeta \\ a \\ b \end{bmatrix} = P \begin{bmatrix} \nu \Delta \varphi - (\nabla v^S)^* \varphi + (v^S \cdot \nabla) \varphi \\ -M^{-1} \displaystyle\int_{\partial \mathcal{S}} 2\nu D(\varphi) n \, d\Gamma - M^{-1} \begin{bmatrix} \displaystyle\int_{\mathcal{F}} \Gamma_4 \cdot \varphi \, dy \\ \displaystyle\int_{\mathcal{F}} \Gamma_5 \cdot \varphi \, dy \end{bmatrix} + M^{-1} a \\ -I^{-1} \displaystyle\int_{\partial \mathcal{S}} y^\perp \cdot 2\nu D(\varphi) n \, d\Gamma - I^{-1} \int_{\mathcal{F}} \Gamma_6 \cdot \varphi \, dy + I^{-1} b \\ - \begin{bmatrix} \displaystyle\int_{\mathcal{F}} \Gamma_1 \cdot \varphi \, dy \\ \displaystyle\int_{\mathcal{F}} \Gamma_2 \cdot \varphi \, dy \end{bmatrix} \\ -(f_M^S)^\perp \cdot \xi - \displaystyle\int_{\mathcal{F}} \Gamma_3 \cdot \varphi \, dy \end{bmatrix}, \tag{3.20}$$

where Γ_i, $i = 1, \ldots, 6$ are the fonctions in $L^\infty(\mathcal{F}; \mathbb{R}^2)$ introduced in (2.29).

Proof. The proof is analogous to the one yielding characterization (3.17), (3.18) when A is defined from the bilinear form $\mathbf{a}(\cdot, \cdot)$, see the proof of Proposition 16. Note that the integration by parts yielding (3.20) uses $v^S = 0$ on $\partial \mathcal{S}$. □

Note that for λ_0 in (3.19) we can define the fractional powers $(\lambda_0 - A)^\alpha$ and $(\lambda_0 - A^*)^\alpha$, $\alpha \in [0, 1]$. We have the following interpolation characterization of their domains.

Proposition 18. *For $\alpha \in [0, 1]$ the following equalities holds*

$$\mathcal{D}((\lambda_0 - A)^\alpha) = [\mathcal{D}(A), \mathcal{H}]_{1-\alpha} = [\mathcal{D}(A^*), \mathcal{H}]_{1-\alpha} = \mathcal{D}((\lambda_0 - A^*)^\alpha), \tag{3.21}$$

where $[\cdot, \cdot]$. denotes the complex interpolation method. Moreover, the following equalities hold

$$\mathcal{D}((\lambda_0 - A)^\alpha) = \mathbf{H}^{2\alpha}(\mathcal{F}) \cap \mathcal{H} \times \mathbb{R}^6 \quad \text{if } \alpha \in \left(0, \frac{1}{4}\right) \tag{3.22}$$

$$\mathcal{D}((\lambda_0 - A)^\alpha) = \Big\{[w, \ell, \omega, h, \theta]^* \in \mathbf{H}^{2\alpha}(\mathcal{F}) \times \mathbb{R}^6, \ w = \ell + \omega y^\perp \text{ on } \partial \mathcal{S}, \ w = 0 \text{ on } \partial \Omega\Big\} \text{ if } \alpha \in \left(\frac{1}{4}, 1\right)$$

and we have in particular $\mathcal{D}((\lambda_0 - A)^{1/2}) = \mathcal{D}((\lambda_0 - A^*)^{1/2}) = \mathcal{V}$.

Proof. Equalities (3.21) are consequences of $\mathcal{D}(A^*) = \mathcal{D}(A)$ and of the maximal accretivity of $\lambda_0 - A$, see [8, Chap. 2, Prop. 6.1]. To prove the last claims we introduce the Dirichlet map $D_0 \in \mathcal{L}(\mathbb{R}^3, \mathbf{H}^2(\mathcal{F}))$ defined by $D_0(\ell, \omega) = z$ where z is the solution of

$$\left\{\begin{array}{rl} -\Delta z + \nabla \pi & = 0 \ \text{ in } \mathcal{F}, \\ \operatorname{div} z & = 0 \ \text{ in } \mathcal{F}, \\ z & = 0 \ \text{ on } \partial\Omega, \\ z & = \ell + \omega y^\perp \ \text{ on } \partial\mathcal{S}. \end{array}\right.$$

It is clear that $\mathcal{D}(A) = \{[w, \ell, \omega, h, \theta]^* \in \mathbf{L}^2(\mathcal{F}) \times \mathbb{R}^6 \mid w - D_0(\ell, \omega) \in \mathbf{V}^2_0(\Omega)\}$ and that $\mathcal{H} = \{[w, \ell, \omega, h, \theta]^* \in \mathbf{L}^2(\mathcal{F}) \times \mathbb{R}^6 \mid w - D_0(\ell, \omega) \in \mathbf{V}^0_n(\mathcal{F})\}$. Then by using the fact that $[w, \ell, \omega, h, \theta]^* \mapsto [w - D_0(\ell, \omega), \ell, \omega, h, \theta]^*$ is an isomorphism from $\mathcal{D}(A)$ onto $\mathbf{V}^2_0(\Omega) \times \mathbb{R}^6$ as well as from $\mathcal{H}$ onto $\mathbf{V}^0_n(\Omega) \times \mathbb{R}^6$, we deduce by interpolation that for all $\alpha \in [0, 1]$:

$$[\mathcal{D}(A), \mathcal{H}]_{1-\alpha} = \{[w, \ell, \omega, h, \theta] \in \mathbf{L}^2(\mathcal{F}) \times \mathbb{R}^6 \mid w - D_0(\ell, \omega) \in [\mathbf{V}^2_0(\mathcal{F}), \mathbf{V}^0_n(\mathcal{F})]_{1-\alpha}\}.$$

Then the conclusion follows from (3.21), from $[\mathbf{V}^2_0(\mathcal{F}), \mathbf{V}^0_n(\mathcal{F})]_{1-\alpha} = [\mathbf{H}^2(\mathcal{F}) \cap \mathbf{H}^1_0(\mathcal{F}), \mathbf{L}^2(\mathcal{F})]_{1-\alpha} \cap \mathbf{V}^0_n(\mathcal{F})$ (see [18]) and from the characterization of this last interpolation space (see [21]). $\square$

Next, we introduce the Dirichlet operator $D_{\mathcal{F}} : \mathbf{V}^0(\partial\Omega) \to \mathbf{L}^2(\mathcal{F}) \times \mathbb{R}^6$ defined as follows: for $u \in \mathbf{V}^0(\partial\Omega)$ we denote by $D_{\mathcal{F}} u \stackrel{\text{def}}{=} [w_u \ \ell_u \ \omega_u \ h_u \ \theta_u]^*$ the unique solution of

$$\left\{\begin{array}{rcl} \lambda_0 w - \nu\Delta w + \Gamma(h, \theta, \ell, \omega) + (w \cdot \nabla) v^S + (v^S \cdot \nabla) w + \nabla q & = & 0 \quad \text{in } \mathcal{F}, \\ \operatorname{div} w & = & 0 \quad \text{in } \mathcal{F}, \\ \\ w & = & \ell + \omega y^\perp \quad \text{on } \partial\mathcal{S}, \\ w & = & u \quad \text{on } \partial\Omega, \\ \lambda_0 M \ell + \displaystyle\int_{\partial\mathcal{S}} \mathbb{T}(w, q) n \, d\Gamma + \theta (f^S_M)^\perp & = & 0, \\ \lambda_0 I \omega + \displaystyle\int_{\partial\mathcal{S}} y^\perp \cdot \mathbb{T}(w, q) n \, d\Gamma & = & 0, \\ \lambda_0 h - \ell & = & 0, \\ \lambda_0 \theta - \omega & = & 0. \end{array}\right.$$

Proposition 19. *The mapping $D_{\mathcal{F}}$ defined above satisfies the following boundedness property:*

$$D_{\mathcal{F}} \in \mathcal{L}(\mathbf{V}^s(\partial\Omega), \mathbf{H}^{s+\frac{1}{2}}(\mathcal{F}) \times \mathbb{R}^6) \quad s \in \left[-\frac{1}{2}, \frac{3}{2}\right]. \tag{3.23}$$

Proof. To obtain (3.23) it suffices to prove it for $s = \frac{3}{2}$ and $s = -\frac{1}{2}$ and then use an interpolation argument. First, the case $s = \frac{3}{2}$ can be obtained from a lifting argument: according to [1, Cor. 3.8] there exists $z \in \mathbf{H}^2(\Omega)$ such that $\operatorname{div} z = 0$ in $\mathcal{F}$, $z = 0$ on $\partial\mathcal{S}$ and $z = u$ on $\partial\Omega$ and satisfying $\|z\|_{\mathbf{H}^2(\Omega)} \leqslant C \|u\|_{\mathbf{V}^{\frac{3}{2}}(\partial\Omega)}$. Then writing $w = \widetilde{w} + z$ we verify that

$$(\lambda_0 - A) \begin{bmatrix} \widetilde{w} \\ \ell \\ \omega \\ h \\ \theta \end{bmatrix} = P \begin{bmatrix} -\lambda_0 z + \nu\Delta z - (z \cdot \nabla) v^S - (v^S \cdot \nabla) z \\ -M^{-1} \displaystyle\int_{\partial\mathcal{S}} 2\nu D(z) n \, d\Gamma \\ -I^{-1} \displaystyle\int_{\partial\mathcal{S}} y^\perp \cdot 2\nu D(z) n \, d\Gamma \\ 0 \\ 0 \end{bmatrix} \in \mathcal{H}.$$

It implies $[\widetilde{w}, \ell, \omega, h, \theta]^* \in \mathcal{D}(A) \subset \mathbf{H}^2(\mathcal{F}) \times \mathbb{R}^6$, from which the case $s = \frac{3}{2}$ is an easy consequence. To prove the case $s = -\frac{1}{2}$, we recall that in that case $D_{\mathcal{F}} u$ is defined by duality as follows: for any $[f, \xi_f, \zeta_f, a_f, b_f]^* \in \mathbf{L}^2(\mathcal{F}) \times \mathbb{R}^6$,

$$\langle D_{\mathcal{F}} u, [f, \xi_f, \zeta_f, a_f, b_f]^* \rangle \stackrel{\text{def}}{=} -\int_{\partial\Omega} u \cdot \mathbb{T}(\varphi, \pi) n d\Gamma, \tag{3.24}$$

where $(\varphi, \pi, \xi, \zeta, a, b)$ is the solution of

$$
\left\{
\begin{aligned}
\lambda_0 \varphi - \nu \Delta \varphi + (\nabla v^S)^* \varphi - (v^S \cdot \nabla)\varphi + \nabla \pi &= f \quad \text{in } \mathcal{F}, \quad \int_{\partial\Omega} \pi d\Gamma = 0, \\
\operatorname{div} \varphi &= 0 \quad \text{in } \mathcal{F} \\
\varphi &= \xi + \zeta y^\perp \quad \text{on } \partial\mathcal{S}, \\
\varphi &= 0 \quad \text{on } \partial\Omega, \\
M\lambda_0 \xi + \int_{\partial\mathcal{S}} \mathbb{T}(\varphi,\pi) n \, d\Gamma + \begin{bmatrix} \displaystyle\int_{\mathcal{F}} \Gamma_4 \cdot \varphi \, dy \\ \displaystyle\int_{\mathcal{F}} \Gamma_5 \cdot \varphi \, dy \end{bmatrix} - a &= \xi_f, \\
I\lambda_0 \zeta + \int_{\partial\mathcal{S}} y^\perp \cdot \mathbb{T}(\varphi,\pi) n \, d\Gamma + \int_{\mathcal{F}} \Gamma_6 \cdot \varphi \, dy - b &= \zeta_f, \\
\lambda_0 a + \begin{bmatrix} \displaystyle\int_{\mathcal{F}} \Gamma_1 \cdot \varphi \, dy \\ \displaystyle\int_{\mathcal{F}} \Gamma_2 \cdot \varphi \, dy \end{bmatrix} &= a_f, \\
\lambda_0 b + (f_M^S)^\perp \cdot \xi + \int_{\mathcal{F}} \Gamma_3 \cdot \varphi \, dy &= b_f.
\end{aligned}
\right.
\tag{3.25}
$$

Note that we chose the normalization condition $\int_{\partial\Omega} \pi d\Gamma = 0$ to guarantee $\int_{\partial\Omega} \mathbb{T}(\varphi,\pi)n \cdot n d\Gamma = 0$ and then $\mathbb{T}(\varphi,\pi)n \in \mathbf{V}^0(\partial\Omega)$.

Since the above system (3.25) can be written as $(\lambda_0 - A^*)[\varphi, \xi, \zeta, a, b]^* = P[f, \xi_f, \zeta_f, b_f, \kappa_f]^*$ we deduce that $[\varphi, \xi, \zeta, b, \kappa]^* \in \mathcal{D}(A^*)$ and $\|[\varphi, \xi, \zeta, b, \kappa]^*\|_{\mathcal{D}(A^*)} \leqslant C\|P[f, \xi_f, \zeta_f, b_f, \kappa_f]^*\|_{\mathcal{H}}$. We deduce that

$$
\|\mathbb{T}(\varphi,\pi)n\|_{\mathbf{V}^{\frac{1}{2}}(\partial\Omega)} \leqslant C\|[f, \xi_f, \zeta_f, b_f, \kappa_f]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^6}
$$

which yields the result for $s = -\frac{1}{2}$. □

Next, we define the input operator

$$
B : \mathbf{V}^0(\partial\Omega) \to [\mathcal{D}(A^*)]', \quad Bu = (\lambda_0 - A)PD_{\mathcal{F}}u. \tag{3.26}
$$

Proposition 20. *The operator B defined by* (3.26) *satisfies:*

$$
(\lambda_0 - A)^{-1+\epsilon} B \in \mathcal{L}(\mathbf{V}^0(\partial\Omega), \mathcal{H}), \quad \epsilon \in \left(0, \frac{1}{4}\right). \tag{3.27}
$$

Moreover, the adjoint of B is defined by

$$
B^* \left[\varphi, \xi, \zeta, a, b\right]^* = -\mathbb{T}(\varphi,\pi)n, \tag{3.28}
$$

where

$$
\begin{bmatrix} \nabla\pi \\ -M^{-1}\displaystyle\int_{\partial\mathcal{S}} \pi n \, d\Gamma \\ -I^{-1}\displaystyle\int_{\partial\mathcal{S}} \pi n \cdot y^\perp \, d\Gamma \\ 0 \\ 0 \end{bmatrix} = (I - P) \begin{bmatrix} -\lambda_0\varphi + \nu\Delta\varphi - (\nabla v^S)^*\varphi + (v^S \cdot \nabla)\varphi \\ -M\lambda_0\xi - \displaystyle\int_{\partial\mathcal{S}} 2\nu D(\varphi) n \, d\Gamma - \begin{bmatrix} \displaystyle\int_{\mathcal{F}} \Gamma_4 \cdot \varphi \, dy \\ \displaystyle\int_{\mathcal{F}} \Gamma_5 \cdot \varphi \, dy \end{bmatrix} + a \\ -I\lambda_0\zeta - \displaystyle\int_{\partial\mathcal{S}} y^\perp \cdot 2\nu D(\varphi) n \, d\Gamma - \int_{\mathcal{F}} \Gamma_6 \cdot \varphi \, dy + b \\ -\lambda_0 a - \begin{bmatrix} \displaystyle\int_{\mathcal{F}} \Gamma_1 \cdot \varphi \, dy \\ \displaystyle\int_{\mathcal{F}} \Gamma_2 \cdot \varphi \, dy \end{bmatrix} \\ -\lambda_0 b - (f_M^S)^\perp \cdot \xi - \displaystyle\int_{\mathcal{F}} \Gamma_3 \cdot \varphi \, dy \end{bmatrix} \tag{3.29}
$$

Proof. The regularity property (3.27) is a direct consequence of (3.15), (3.22), (3.23) and the characterization (3.28) follows directly from (3.24). $\square$

We are now in position to deduce formulation (3.11)-(3.12) from (3.2)-(3.9) and (3.10). If we suppose that $(w, \ell, \omega, a, \theta)$ is a regular solution of (3.2)-(3.9) then by multiplying equalities (3.2), (3.6), (3.7), (3.8), (3.9) by the components of $\mathbf{Y} = [\varphi, \xi, \zeta, a, b]^* \in \mathcal{D}(A^*)$ and integrating by parts, after some calculations we obtain that $\mathbf{X} = [w, \ell, \omega, h, \theta]^*$ satisfies for $t \geqslant 0$:

$$\frac{d}{dt}\langle P\mathbf{X}(t), \mathbf{Y}\rangle - \langle P\mathbf{X}(t), A^*\mathbf{Y}\rangle - \langle \mathbf{F}, \mathbf{Y}\rangle = -\int_{\partial\Omega} u(t) \cdot \mathbb{T}(\varphi, \pi) n d\Gamma = \langle u(t), B^*\mathbf{Y}\rangle.$$

The fact that the above equality is satisfied for all $\mathbf{Y} \in \mathcal{D}(A^*)$ exactly means that $P\mathbf{X}$ satisfies the first equality of (3.11) in $[\mathcal{D}(A^*)]'$. Moreover, since we have $\mathbf{X} - D_{\mathcal{F}}u \in \mathcal{H}$ we have $(I - P)(\mathbf{X} - D_{\mathcal{F}}u) = 0$ which implies that $\mathbf{X}$ satisfies (3.12).

3.2 Stabilizability of the linear system

The goal of this subsection is to prove, for a fixed rate of decrease $\sigma > 0$, the existence of a feedback control

$$u(t) = \sum_{j=1}^{N_\sigma} \left(\int_{\mathcal{F}} w(t) \cdot \varphi_j dy + M\ell(t) \cdot \xi_j + I\omega(t)\zeta_j + h(t) \cdot a_j + \theta(t) b_j \right) v_j \tag{3.30}$$

such that the solution of (3.2)-(3.9) with $(F, \ell_F, \omega_F, h_F, \theta_F) = (0, 0, 0, 0, 0)$ tends to zero as $t \to +\infty$ with an exponential rate of decrease $\sigma > 0$. For that, we are going to show the existence of families $(\varphi_j, \xi_j, \zeta_j, a_j, b_j)$ and v_j, $j = 1, \ldots, N_\sigma$ such that the underlying closed-loop linear operator of (3.2)-(3.9) with (3.30) generates and analytic and exponentially stable semigroup of type lower than $-\sigma$ (see [8, II-1, Cor. 2.1]). It then permits to deduce results for the case of non zero $(F, \ell_F, \omega_F, h_F, \theta_F)$ that are used in the next subsection to construct fixed-point solutions of the nonlinear system (2.30)-(2.38).

Proposition 21. *For $\sigma > 0$, there exist $N_\sigma \in \mathbb{N}^*$ and families $[\varphi_j, \xi_j, \zeta_j, a_j, b_j]^* \in \mathcal{D}(A^*)$ and $v_j \in \mathbf{V}^{\frac{3}{2}}(\partial\Omega)$, $j = 1, \ldots, N_\sigma$, and a corresponding feedback operator $F_\sigma : \mathcal{H} \to \mathbf{V}^{\frac{3}{2}}(\partial\Omega)$ defined by*

$$F_\sigma[w, \ell, \omega, h, \theta]^* = \sum_{j=1}^{N_\sigma} \left(\int_{\mathcal{F}} w \cdot \varphi_j dy + M\ell \cdot \xi_j + I\omega\zeta_j + h \cdot a_j + \theta b_j \right) v_j \tag{3.31}$$

such that the linear operator $A_\sigma \stackrel{def}{=} A + BF_\sigma$ with domain $\mathcal{D}(A_\sigma) \stackrel{def}{=} \{\mathbf{X} \in \mathcal{H} \mid A\mathbf{X} + BF_\sigma\mathbf{X} \in \mathcal{H}\}$ is the infinitesimal generator of an analytic and exponentially stable semigroup on $\mathcal{H}$ of type lower than $-\sigma$.

Proof. The proof of the above proposition relies on the Hautus-Fattorini stabilizability criterion, see [5, Theorem 1] or [7]. Since A has compact resolvent and generates an analytic semigroup on $\mathcal{H}$, and since B is relatively bounded with respect to A, then the homogeneous linear system is stabilizable by finite dimensional feedback control for any rate of decrease if and only if the following criterion is satisfied for all $\lambda \in \mathbb{C}$:

$$\lambda\mathbf{Y} - A^*\mathbf{Y} = 0 \quad \text{and} \quad B^*\mathbf{Y} = 0 \quad \Longrightarrow \quad \mathbf{Y} = 0. \tag{3.32}$$

From (3.20) and (3.28) we deduce that (3.32) is true if and only if all $[\varphi,\xi,\zeta,b,\kappa]^* \in \mathcal{D}(A^*)$ which satisfies

$$\left\{\begin{array}{rcl}
\lambda\varphi - \nu\Delta\varphi + (\nabla v^S)^*\varphi - (v^S\cdot\nabla)\varphi + \nabla\pi & = & 0 \quad \text{in } \mathcal{F}, \quad \displaystyle\int_{\partial\Omega}\pi d\Gamma = 0,\\
\operatorname{div}\varphi & = & 0 \quad \text{in } \mathcal{F},\\
\varphi & = & \xi + \zeta y^\perp \quad \text{on } \partial\mathcal{S},\\
\varphi & = & 0 \quad \text{on } \partial\Omega,\\
M\lambda\xi + \displaystyle\int_{\partial\mathcal{S}}\mathbb{T}(\varphi,\pi)n\; d\Gamma + \begin{bmatrix}\displaystyle\int_{\mathcal{F}}\Gamma_4\cdot\varphi\; dy\\ \displaystyle\int_{\mathcal{F}}\Gamma_5\cdot\varphi\; dy\end{bmatrix} - a & = & 0,\\
I\lambda\zeta + \displaystyle\int_{\partial\mathcal{S}} y^\perp\cdot\mathbb{T}(\varphi,\pi)n\; d\Gamma + \displaystyle\int_{\mathcal{F}}\Gamma_6\cdot\varphi\; dy - b & = & 0,\\
\lambda a + \begin{bmatrix}\displaystyle\int_{\mathcal{F}}\Gamma_1\cdot\varphi\; dy\\ \displaystyle\int_{\mathcal{F}}\Gamma_2\cdot\varphi\; dy\end{bmatrix} & = & 0,\\
\lambda b + (f_M^S)^\perp\cdot\xi + \displaystyle\int_{\mathcal{F}}\Gamma_3\cdot\varphi\; dy & = & 0.
\end{array}\right. \tag{3.33}$$

and

$$\mathbb{T}(\varphi,\pi)n = 0 \quad \text{on } \partial\Omega. \tag{3.34}$$

must be identically zero.

The above implication can be proved as follows: combining (3.34) and a classical uniqueness result for Stokes type systems (see e.g. [16]) we deduce that $\varphi = 0$ and $\pi = 0$ on $\mathcal{F}$. Then, since $\varphi = \xi + \zeta y^\perp$ on $\mathcal{S}$, we deduce that $\xi = 0$ and $\zeta = 0$. Coming back to (3.33) we deduce $a = 0$ and $b = 0$.

Then the general framework of [5, 7] can be applied and for a given $\sigma > 0$, there exist families

$$(\varphi_j,\xi_j,\zeta_j,a_j,b_j) \in \mathcal{D}(A^*)$$

and $v_j \in \mathbf{V}^0(\partial\Omega)$, $j = 1,\dots,N_\sigma$, and a feedback control of the form (3.31) such that the conclusions of the proposition hold. Moreover, each v_j can be chosen in $\mathbf{V}^{\frac{3}{2}}(\partial\Omega)$. This comes from the fact that the set of admissible families (v_j) is a nonempty open set of $(\mathbf{V}^0(\partial\Omega))^{N_\sigma}$ (see [5, Theorem 5] or [7, Theorem 6]). Indeed, if a family $(\tilde v_j)$ is admissible then all families in a neighborhood of $(\tilde v_j)$ in $(\mathbf{V}^0(\partial\Omega))^{N_\sigma}$ are admissible. Then the conclusion follows from the density of $\mathbf{V}^{\frac{3}{2}}(\partial\Omega)$ in $\mathbf{V}^0(\partial\Omega)$.

□

Remark 22. *There is a wide choice for the family (v_j) in (3.31) since it is proved in [5, 7] that a family (v_j) is generically admissible provided that N_σ is greater of equal to the maximum of the geometric multiplicities of eigenvalues with real part greater than $-\sigma$. However, a practical choice could be the range through B^* of real and imaginary parts of eigenvectors corresponding to above mentioned eigenvalues. Once the family (v_j) is determined, the family $(\varphi_j,\xi_j,\zeta_j,a_j,b_j)$ in (3.31) can be obtained for instance from the solution of a finite dimensional Riccati equation of size $M_\sigma\times M_\sigma$, where M_σ is the dimension of the subspace composed with eigenvectors related to eigenvalues with real part greater than $-\sigma$. We refer to [5, 7] or to [29] for details.*

Proposition 23. *For $\alpha\in[0,1]$ we have $\mathcal{D}((-A_\sigma)^\alpha) \hookrightarrow [\mathbf{H}^{2\alpha}(\mathcal{F})\times\mathbb{R}^6]\cap\mathcal{H}$ and $\mathcal{D}((-A_\sigma^*)^\alpha) = \mathcal{D}((\lambda_0 - A^*)^\alpha)$.*

Proof. First, since (3.27) implies that $F_\sigma^* B^*$ is relatively bounded with respect to A^*, the equality $\mathcal{D}(A_\sigma^*) = \mathcal{D}(A^*)$ follows from the expression $A_\sigma^* = A^* + F_\sigma^* B^*$ with a perturbation argument. Thus, since the interpolation equality $\mathcal{D}((\lambda_0 - A^*)^\alpha) = [\mathcal{D}(A^*),\mathcal{H}]_{1-\alpha}$ holds for all $\alpha\in[0,1]$ a perturbation argument (see [13, Prop. 2.7]) yields that the analogous equality $\mathcal{D}((-A_\sigma^*)^\alpha) = [\mathcal{D}(A_\sigma^*),\mathcal{H}]_{1-\alpha}$ is also true for all $\alpha\in[0,1]$. Then $\mathcal{D}((-A_\sigma^*)^\alpha) = \mathcal{D}((\lambda_0 - A^*)^\alpha)$ follows from $\mathcal{D}(A_\sigma^*) = \mathcal{D}(A^*)$. Moreover, by duality we also have $\mathcal{D}((-A_\sigma)^\alpha) = [\mathcal{D}(A_\sigma),\mathcal{H}]_{1-\alpha}$ for all $\alpha\in[0,1]$ (see [8, II. Thm. 6.1 (iv)]). Then it remains to prove $\mathcal{D}(A_\sigma) \hookrightarrow (\mathbf{H}^2(\mathcal{F})\times\mathbb{R}^6)\cap\mathcal{H}$ to deduce $\mathcal{D}((-A_\sigma)^\alpha) \hookrightarrow (\mathbf{H}^{2\alpha}(\mathcal{F})\times\mathbb{R}^6)\cap\mathcal{H}$ for $\alpha\in[0,1]$ by interpolation.

To show $\mathcal{D}(A_\sigma) \hookrightarrow (\mathbf{H}^2(\mathcal{F})\times\mathbb{R}^6)\cap\mathcal{H}$ we come back to the definitions $\mathcal{D}(A_\sigma) \stackrel{\text{def}}{=} \{\mathbf{X}\in\mathcal{H} \mid A\mathbf{X}+BF_\sigma\mathbf{X}\in\mathcal{H}\}$ and (3.26) from which we deduce

$$\mathcal{D}(A_\sigma) = \{\mathbf{X}\in\mathcal{H} \mid \mathbf{X} - PDF_\sigma\mathbf{X}\in\mathcal{D}(A)\}.$$

Then since each v_j belongs to $\mathbf{V}^{\frac{3}{2}}(\partial\Omega)$, with (3.31), (3.23) and (3.15) we deduce that $PDF_\sigma\mathbf{X}\in(\mathbf{H}^2(\mathcal{F})\times\mathbb{R}^6)\cap\mathcal{H}$ if $\mathbf{X}\in\mathcal{H}$ and with (3.17) it yields $\mathcal{D}(A_\sigma)\hookrightarrow\mathbf{H}^2(\mathcal{F})\times\mathbb{R}^6$. □

Finally, the system (3.11), (3.12) with the feedback control $u = F_\sigma P\mathbf{X}$ can be rewritten

$$\begin{aligned} P\mathbf{X}' &= A_\sigma P\mathbf{X} + P\mathbf{F} \ \text{ in } [\mathcal{D}(A^*)]', \quad P\mathbf{X}(0) = P\mathbf{X}^0 &\quad (3.35)\\ (I-P)\mathbf{X} &= (I-P)D_{\mathcal{F}}F_\sigma P\mathbf{X}. &\quad (3.36)\end{aligned}$$

Let us state regularity results for system (3.35), (3.36). For that we introduce the notation

$$\mathcal{V}_\sigma \stackrel{\text{def}}{=} \mathcal{D}((-A_\sigma)^{1/2}), \tag{3.37}$$

and, according to Proposition 23, we note that $\mathcal{V}_\sigma \hookrightarrow (\mathbf{H}^1(\mathcal{F})\times\mathbb{R}^6)\cap\mathcal{H}$ and that $(\mathcal{V}_\sigma)' = (\mathcal{V})'$. We also introduce a smooth nonnegative function $\chi:\mathbb{R}^+\to\mathbb{R}$ satisfying

$$\chi\in C^\infty(\mathbb{R}^+), \qquad \begin{cases} \chi(t)\in[0,1] & \forall t\in\mathbb{R}^+\\ \chi(t) = t & \forall t\in[0,\frac{1}{2}]\\ \chi(t)=1 & \forall t\in[1,\infty)\end{cases} \tag{3.38}$$

Let us recall that the spaces $W_\sigma(\cdot,\cdot)$, $L^2_\sigma(\cdot)$, $H^1_\sigma(\cdot)$ below are defined by (3.1) from the spaces $W(\cdot,\cdot)$, $L^2(\cdot)$, $H^1(\cdot)$.

Proposition 24. *Let χ be a function satisfying (3.38). Assume that $\mathbf{X}^0\in\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^6$ and $P\mathbf{F}\in L^2_\sigma(\mathcal{V}')$ with $\chi P\mathbf{F}\in L^2_\sigma(\mathcal{H})$. Then system (3.35), (3.36) admits a unique solution*

$$\mathbf{X}\in W_\sigma(\mathcal{V}_\sigma,\mathcal{V}') + H^1_\sigma(\mathbf{H}^2(\mathcal{F})\times\mathbb{R}^6).$$

Moreover, $\chi\mathbf{X}$ belongs to $W_\sigma(\mathcal{D}(A_\sigma),\mathcal{H}) + H^1_\sigma(\mathbf{H}^2(\mathcal{F})\times\mathbb{R}^6)$ and the following estimate holds

$$\begin{aligned}\|P\mathbf{X}\|_{W_\sigma(\mathcal{V}_\sigma,\mathcal{V}')} + &\|\chi P\mathbf{X}\|_{W_\sigma(\mathcal{D}(A_\sigma),\mathcal{H})} + \|(I-P)\mathbf{X}\|_{H^1_\sigma(\mathbf{H}^2(\mathcal{F})\times\mathbb{R}^6)}\\ &\leqslant C(\|P\mathbf{F}\|_{L^2_\sigma(\mathcal{V}')} + \|\chi P\mathbf{F}\|_{L^2_\sigma(\mathcal{H})} + \|P\mathbf{X}^0\|_{\mathcal{H}}).\end{aligned}$$

Proof. By construction, A_σ is the generator of an analytic semigroup of type $-\sigma$. Therefore, using classical maximal regularity result for analytic semigroup (see [8, Theorem 2.2, p.208]), we deduce from $P\mathbf{X}^0\in\mathcal{H}$, $P\mathbf{F}\in L^2_\sigma(\mathcal{V}')$ that (3.35) admits a unique solution $P\mathbf{X}\in W_\sigma(\mathcal{V}_\sigma,\mathcal{V}')$ and that

$$\|P\mathbf{X}\|_{W_\sigma(\mathcal{V}_\sigma,\mathcal{V}')} \leqslant C(\|P\mathbf{X}^0\|_{\mathcal{H}} + \|P\mathbf{F}\|_{L^2_\sigma(\mathcal{V}')}). \tag{3.39}$$

Moreover, since $\chi P\mathbf{X}$ satisfies

$$(\chi P\mathbf{X})' = A_\sigma(\chi P\mathbf{X}) + \chi P\mathbf{F} + \chi' P\mathbf{X}, \quad (\chi P\mathbf{X})(0) = 0$$

we deduce from $\chi P\mathbf{F}\in L^2_\sigma(\mathcal{H})$ that

$$\|\chi P\mathbf{X}\|_{W_\sigma(\mathcal{D}(A_\sigma),\mathcal{H})} \leqslant C(\|\chi P\mathbf{F}\|_{L^2_\sigma(\mathcal{H})} + \|P\mathbf{X}\|_{L^2_\sigma(\mathcal{H})}). \tag{3.40}$$

Finally, since $[\varphi_j,\xi_j,\zeta_j,a_j,b_j]^*\in\mathcal{D}(A^*)$, the definition (3.31) of F_σ and the fact that $\operatorname{ran} F_\sigma\subset\mathbf{V}^{3/2}(\partial\Omega)$ imply that it can be extended to an operator in $\mathcal{L}(\mathcal{V}',\mathbf{V}^{3/2}(\partial\Omega))$. Therefore, using (3.36), (3.23), (3.15), and the above remark, we deduce that

$$\|(I-P)\mathbf{X}\|_{H^1_\sigma(\mathbf{H}^2(\mathcal{F})\times\mathbb{R}^6)} \leqslant C\|P\mathbf{X}\|_{H^1_\sigma(\mathcal{V}')}. \tag{3.41}$$

Then the conclusion follows by combining (3.39), (3.40) and (3.41). □

Next, we deduce regularity results for the system (3.2)–(3.9).

Corollary 25. *Let χ be a function satisfying* (3.38). *Assume $[w^0,\ell^0,\omega^0,h^0,\theta^0]^* \in \mathbf{L}^2(\mathcal{F})\times\mathbb{R}^6$ and that $\mathbf{F}=[F,\ell_F,\omega_F,h_F,\theta_F]^*$ satisfies $P\mathbf{F}\in L^2_\sigma(\mathcal{V}')$ and $\chi\mathbf{F}\in L^2_\sigma(\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^6)$. Then the system* (3.2)–(3.9), (3.10) *and* (3.30) *admits a unique solution*

$$(w,\ell,\omega,h,\theta)\in\left[L^2_\sigma(\mathbf{H}^1(\mathcal{F}))\cap C_{b,\sigma}(\mathbf{L}^2(\mathcal{F}))\right]\times(C_{b,\sigma})^6,$$
$$\chi(w,q,\ell,\omega,h,\theta)\in\left[W_\sigma(\mathbf{H}^2(\mathcal{F}),\mathbf{L}^2(\mathcal{F}))\right]\times L^2_\sigma(H^1(\mathcal{F}))\times(H^1_\sigma)^6.$$

This solution satisfies

$$\begin{aligned}\|w\|_{L^2_\sigma(\mathbf{H}^1(\mathcal{F}))}+&\|w\|_{L^\infty_\sigma(\mathbf{L}^2(\mathcal{F}))}+\|(\ell,\omega,h,\theta)\|_{(L^\infty_\sigma)^6}+\|\chi w\|_{W_\sigma(\mathbf{H}^2(\mathcal{F}),\mathbf{L}^2(\mathcal{F}))}+\|\chi(\ell,\omega,h,\theta)\|_{(H^1_\sigma)^6}\\ &+\|\chi q\|_{L^2_\sigma(H^1(\mathcal{F}))}\leqslant C(\|P\mathbf{F}\|_{L^2_\sigma(\mathcal{V}')}+\|\chi\mathbf{F}\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^6)}+\|P\mathbf{X}^0\|_{\mathcal{H}}).\end{aligned}\tag{3.42}$$

Moreover, if $h^0=0$, $\theta^0=0$ and if $h_F\in(L^\infty)^2$, $\theta_F\in L^\infty$ then

$$\begin{aligned}|h(t)|+|\theta(t)|&\leqslant\chi(t)C\left(\|P\mathbf{F}\|_{L^2_\sigma(\mathcal{V}')}+\|\chi\mathbf{F}\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^6)}+\|P\mathbf{X}^0\|_{\mathcal{H}}+\|(h_F,\theta_F)\|_{(L^\infty)^3}\right),\\ |h'(t)|+|\theta'(t)|&\leqslant C\left(\|P\mathbf{F}\|_{L^2_\sigma(\mathcal{V}')}+\|\chi\mathbf{F}\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^6)}+\|P\mathbf{X}^0\|_{\mathcal{H}}+\|(h_F,\theta_F)\|_{(L^\infty)^3}\right).\end{aligned}\tag{3.43}$$

Proof. First, we notice that system (3.2)–(3.9) and (3.10) can be written as system (3.35), (3.36), with A_σ, F_σ defined in Proposition 21 and the other operators defined in Subsection 3.1. In particular, we deduce from Proposition 24 that

$$\mathbf{X}=[w,\ell,\omega,h,\theta]^*\in W_\sigma(\mathcal{D}(\mathcal{V}_\sigma),\mathcal{V}')+H^1_\sigma(\mathbf{H}^2(\mathcal{F})\times\mathbb{R}^6)\tag{3.44}$$

and

$$\chi\mathbf{X}=\chi[w,\ell,\omega,h,\theta]^*\in W_\sigma(\mathcal{D}(A_\sigma),\mathcal{H})+H^1_\sigma(\mathbf{H}^2(\mathcal{F})\times\mathbb{R}^6).\tag{3.45}$$

Using that $\mathcal{V}_\sigma\hookrightarrow\mathbf{H}^1(\mathcal{F})\times\mathbb{R}^6$ and $W_\sigma(\mathcal{V}_\sigma,\mathcal{V}')\hookrightarrow C_{b,\sigma}(\mathcal{H})$, we deduce that

$$W_\sigma(\mathcal{V}_\sigma,\mathcal{V}')\hookrightarrow L^2_\sigma(\mathbf{H}^1(\mathcal{F})\times\mathbb{R}^6)\cap C_{b,\sigma}(\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^6).$$

The above embedding combined with (3.44) gives

$$w\in L^2_\sigma(\mathbf{H}^1(\mathcal{F}))\cap C_{b,\sigma}(\mathbf{L}^2(\mathcal{F})),\quad[\ell,\omega,h,\theta]^*\in(C_{b,\sigma})^6.$$

In a similar way, we deduce from (3.45) that

$$\chi w\in W_\sigma(\mathbf{H}^2(\mathcal{F}),\mathbf{L}^2(\mathcal{F})),\quad\chi[\ell,\omega,h,\theta]^*\in\left(H^1_\sigma\right)^6.$$

Recovering the pressure function q by using the De Rahm Lemma, we deduce that

$$\chi\nabla q\in L^2_\sigma(\mathbf{L}^2(\mathcal{F})),$$

and its bound (3.42) follows from the estimates on $[w,\ell,\omega,h,\theta]^*$ and from (3.2).

Finally, if $h(0)=0$ and $\theta(0)=0$, we deduce that

$$h(t)=\int_0^t(\ell+h_F)\,ds,\quad\theta(t)=\int_0^t(\omega+\theta_F)\,ds,$$

and if $a_F\in(L^\infty)^2$, $\theta_F\in L^\infty$, the above relations yield

$$|h(t)|+|\theta(t)|\leqslant t(\|\ell\|_{(L^\infty)^2}+\|\omega\|_{L^\infty}+\|(h_F,\theta_F)\|_{(L^\infty)^3}).$$

The above equation and (3.42) yield the first inequality in (3.43). The second inequality in (3.43) is an obvious consequence of $h'=\ell+h_F$ and $\theta'=\omega+\theta_F$ with the bound (3.42). $\square$

3.3 The fixed point procedure

In this subsection, we prove an existence result (Theorem 26 below) for system (2.30)-(2.38) with a feedback control u given by (3.30).

Theorem 26. *Assume* $[w^0,\ell^0,\omega^0,h^0,\theta^0]\in \mathbf{L}^2(\mathcal{F})\times\mathbb{R}^6$ *and* $h^0=0$ *and* $\theta^0=0$. *There exist* $\mu>0$ *and* $\rho>0$ *such that if*

$$\|w^0\|_{\mathbf{L}^2(\mathcal{F})}+|\ell^0|+|\omega^0|\leqslant\mu,$$

then the system (2.30)-(2.38) *with the feedback control* (3.30) *admits a solution in the space*

$$\mathcal{G}\stackrel{def}{=}\Big\{[w,\ell,\omega,h,\theta,q]^*\ ;\quad \chi[w,\ell,\omega,h,\theta,q]^*\in W_\sigma(\mathbf{H}^2(\mathcal{F})\times\mathbb{R}^6,\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^6)\times L^2(H^1(\mathcal{F})),$$
$$\textit{and}\quad [w,\ell,\omega,h,\theta]^*\in L^2_\sigma(\mathbf{H}^1(\mathcal{F})\times\mathbb{R}^6)\cap C_{b,\sigma}(\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^6)\Big\}.$$

In order to prove Theorem 26, we consider the Banach space

$$\mathcal{E}\stackrel{\text{def}}{=}\Big\{\mathbf{F}=[F,\ell_F,\omega_F,h_F,\theta_F]^*\ ;\ P\mathbf{F}\in L^2_\sigma(\mathcal{V}'),\ \chi\mathbf{F}\in L^2_\sigma(\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^6),\ (h_F,\theta_F)\in(L^\infty)^3\Big\}$$

and the following mapping defined on a closed ball of $\mathcal{E}$:

$$\Psi: B_\mathcal{E}(0,\delta)\to\mathcal{E},\quad \mathbf{F}=[F,\ell_F,\omega_F,h_F,\theta_F]^*\mapsto[F(\mathbf{X},q),\varepsilon_\ell(\mathbf{X}),0,\varepsilon_h(\mathbf{X}),0]^*,$$

where $\mathbf{X}\stackrel{\text{def}}{=}[w,\ell,\omega,h,\theta]^*$ and $(w,q,\ell,\omega,h,\theta)\in\mathcal{G}$ is the solution defined in Corollary 25, and where $F(\mathbf{X},q)$, $\varepsilon_\ell(\mathbf{X})$, $\varepsilon_h(\mathbf{X})$ are defined by (2.40) and (2.39).

We remark that if $\mathbf{F}=[F,\ell_F,\omega_F,h_F,\theta_F]^*$ is a fixed point of the mapping Ψ, then the corresponding solution $(w,q,\ell,\omega,h,\theta)$ of (3.2)–(3.9), (3.10) and (3.30) is a solution of (2.30)–(2.38). Consequently, we are reduced to show that Ψ admits a fixed point. We prove that for δ small enough, Ψ is well-defined from $B_\mathcal{E}(0,\delta)$ onto itself and that the restriction of Ψ on this closed ball is a contraction mapping.

First, we notice that (3.43) implies (2.4) provided that $\mathbf{F}\in B_\mathcal{E}(0,\delta)$ with δ small enough and that $\mathbf{X}^0=[w^0,\ell^0,\omega^0,0,0]^*$ has a norm small enough in $\mathcal{H}$. In particular, the changes of variables X and Y are well-defined as well as $F(\mathbf{X},q)$, $\varepsilon_\ell(\mathbf{X})$, and $\varepsilon_h(\mathbf{X})$.

Second, we use several technical results whose proofs are given in Section 4. To simplify the notation, in what follows, we assume

$$\delta+\|[w^0,\ell^0,\omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3}\leqslant 1.$$

Proposition 27. *There exists* $C_\#>0$ *such that for all* $\mathbf{F}\in B_\mathcal{E}(0,\delta)$, *the solution* $(w,q,\ell,\omega,h,\theta)$ *of* (3.2)–(3.9), (3.10) *and* (3.30) *associated with* $\mathbf{F}$ *satisfies*

$$\|P\Psi(\mathbf{F})\|_{L^2_\sigma(\mathcal{V}')}+\|\chi\Psi(\mathbf{F})\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^6)}+\|\varepsilon_h(\mathbf{X})\|_{(L^\infty)^2}\leqslant C_\#(\delta+\|[w^0,\ell^0,\omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3})^2.$$

From the above proposition, we remark that if

$$\|[w^0,\ell^0,\omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3}\leqslant\delta, \tag{3.46}$$

and δ is small enough so that

$$4C_\#\delta\leqslant 1, \tag{3.47}$$

then Ψ is well-defined from $B_\mathcal{E}(0,\delta)$ onto itself.

The second important technical result we need is the following:

Proposition 28. *There exists a positive constant* $C_\#$ *such that for all* $\mathbf{F}^{(1)},\mathbf{F}^{(2)}\in B_\mathcal{E}(0,\delta)$, *the solutions* $(w^{(i)},q^{(i)},\ell^{(i)},\omega^{(i)},h^{(i)},\theta^{(i)})$ *of* (3.2)–(3.9), (3.10) *and* (3.30) *associated with* $\mathbf{F}^{(i)}$ *for* $i=1,2$ *satisfy*

$$\|P(\Psi(\mathbf{F}^{(1)})-\Psi(\mathbf{F}^{(2)}))\|_{L^2_\sigma(\mathcal{V}')}+\|\chi(\Psi(\mathbf{F}^{(1)})-\Psi(\mathbf{F}^{(2)}))\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^6)}+\|\varepsilon_h(\mathbf{X}^{(1)})-\varepsilon_h(\mathbf{X}^{(2)})\|_{(L^\infty)^2}$$
$$\leqslant C_\#(\delta+\|[w^0,\ell^0,\omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3})\|\mathbf{F}^{(1)}-\mathbf{F}^{(2)}\|_\mathcal{E}.$$

With the same conditions (3.46) and (3.47), we deduce that the restriction of Ψ on $B_{\mathcal{E}}(0,\delta)$ is a contraction mapping. The classical Banach fixed point theorem allows to conclude.

Finally, let us remind that Theorem 26 yields Theorem 2 and, in particular, that (1.16) is satisfied. Indeed, from Theorem 26, we deduce that the stabilized solution $[w,\ell,\omega,h,\theta,q]$ satisfies

$$\|w(t)\|_{\mathbf{L}^2(\mathcal{F})} + |\ell(t)| + |\omega(t)| + |h(t)| + |\theta(t)| \leqslant C(\|w^0\|_{\mathbf{L}^2(\mathcal{F})} + |\ell^0| + |\omega^0|)\ e^{-\sigma t}. \tag{3.48}$$

By considering $\widetilde{v}$ defined in Ω by formula (2.5) and with v extended by the rigid velocity $\ell + \omega y^{\perp}$ in $\mathcal{S}$, and by also considering v^S defined in Ω by extending it by zero in $\mathcal{S}$, we can also assume that

$$w(t,y) = \widetilde{v}(t,y) - v^S(y) = \ell(t) + \omega(t)y^{\perp} \quad (y \in \mathcal{S}).$$

Thus some calculation shows that (3.48) implies

$$\|w(t)\|_{\mathbf{L}^2(\Omega)} + |h(t)| + |\theta(t)| \leqslant C\|w^0\|_{\mathbf{L}^2(\Omega)}\ e^{-\sigma t}. \tag{3.49}$$

Then, we can write

$$\begin{aligned}\|v(t) - v^S\|_{\mathbf{L}^2(\Omega)} &\leqslant C\|v(t,X(t,\cdot)) - v^S(X(t,\cdot))\|_{\mathbf{L}^2(\Omega)} \leqslant \\ &C\|\operatorname{Cof}(\nabla Y)(t,\cdot) - I_2\|_{\mathbf{L}^\infty(\Omega)}\|\widetilde{v}(t)\|_{\mathbf{L}^2(\Omega)} + \|w(t)\|_{\mathbf{L}^2(\Omega)} + \|X(t,\cdot) - \operatorname{Id}\|_{\mathbf{L}^\infty(\Omega)}\|\nabla v^S\|_{\mathbf{L}^2(\Omega)}.\end{aligned}$$

Consequently, using the estimates in Section 4 that follows, we deduce

$$\|v(t) - v^S\|_{\mathbf{L}^2(\Omega)} \leqslant C(|h(t)| + |\theta(t)|)(\|v^S\|_{\mathbf{H}^1(\mathcal{F})} + 1) + \|w(t)\|_{\mathbf{L}^2(\Omega)}.$$

The proof concludes by combining the above estimate with (3.49).

Remark 29. *From the use of the Banach fixed point theorem we also have the uniqueness of the solution $(w,q,\ell,\omega,h,\theta)$ within the class of functions belonging to a neighborhood of the origin of $\mathcal{G}$. However, the uniqueness within a class of arbitrary large (and not necessarily stable) functions is not given by Theorem 26.*

Remark 30. *It is important to notice that in Propositions 27 and 28, we use in a crucial way the fact that $h^0 = 0$ and $\theta^0 = 0$. This allows to obtain estimates in Lemma 31 below for the terms $[\mathbf{L} - \Delta]w$, $[\mathbf{K} - \operatorname{Id}]\partial_t w$ and $[\mathbf{G} - \nabla]q$ of $F(\mathbf{X}, q)$. The fact that $h^0 = 0$ and $\theta^0 = 0$ implies that $\mathbf{L} - \Delta$, $\mathbf{K} - \operatorname{Id}$ and $\mathbf{G} - \nabla$ are of order t and suitable estimates are obtained in terms of time L^2 norms of tD^2w, $t\partial_t w$ and tp. It is a key point since for initial velocity in $\mathbf{L}^2(\mathcal{F})$, p and $\partial_t w$ are not L^2-integrable in time.*

4 Estimates of the coefficients

The aim of this section is to prove Proposition 12, Proposition 13, Proposition 27 and Proposition 28. We start by proving Propositions 12 and 13.

In this section C denotes a generic positive constant that may change from line to line and that is independent on a, θ, ℓ, ω and on the variables y, x, but that may depend on $\|\eta\|_{W^{1,\infty}(\Omega)}$ or on the geometry.

4.1 Proof of Propositions 12 and 13

First we write

$$R_\theta - I_2 = \begin{bmatrix} \cos(\theta) - 1 & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) - 1 \end{bmatrix}$$

to obtain

$$\left| R_\theta - I_2 - \theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right| \leqslant C|\theta|^2.$$

In particular, there exists $\varepsilon_1 \in \mathbf{C}^\infty(\mathbb{R}^2 \times \Omega)$ such that

$$X(h,\theta,y) = y + \eta(y)\left[h + \theta y^{\perp}\right] + \varepsilon_1(y,\theta), \quad \text{with} \quad \|\varepsilon_1(\cdot,\theta)\|_{\mathbf{L}^\infty(\Omega)} \leqslant C|\theta|^2. \tag{4.1}$$

We also have

$$\nabla X(h,\theta,y) = I_2 + [h + (R_\theta - I_2)y] \otimes \nabla\eta(y) + \eta(y)(R_\theta - I_2)$$

and thus

$$\nabla X(h,\theta,y) = I_2 + \left[h + \theta y^\perp\right] \otimes \nabla\eta(y) + \eta(y)\theta R_{\pi/2} + \varepsilon_2(y,\theta), \tag{4.2}$$

with

$$\|\varepsilon_2(\cdot,\theta)\|_{\mathbf{L}^\infty(\Omega)} \leqslant C|\theta|^2.$$

We write

$$\mathbb{S}(h,\theta,y) = [h + (R_\theta - I_2)y] \otimes \nabla\eta(y) + \eta(y)(R_\theta - I_2).$$

Assuming (2.4), with $C_\star$ small enough, we obtain that the inverse of ∇X is given by

$$(\nabla X(h,\theta,y))^{-1} = \sum_{k\geqslant 0}(-1)^k \mathbb{S}(h,\theta,y)^k,$$

and thus

$$(\nabla X(h,\theta,y))^{-1} = \nabla Y(h,\theta,X(h,\theta,y)) = I_2 - \left[h + \theta y^\perp\right] \otimes \nabla\eta(y) - \eta(y)\theta R_{\pi/2} + \varepsilon_3(y,h,\theta), \tag{4.3}$$

where

$$\varepsilon_3(y,h,\theta) = \left(\sum_{k\geqslant 0}(-1)^k \mathbb{S}(h,\theta,y)^k\right)\mathbb{S}(h,\theta,y)^2.$$

From above equalities we deduce that

$$\nabla Y(\cdot,X) = I_2 + \Lambda_3(h,\theta) + \varepsilon_3(h,\theta), \tag{4.4}$$

where $\Lambda_3 = (\Lambda_{3ij})$ is a smooth mapping from $\overline{\Omega}$ into $\mathcal{L}(\mathbb{R}^3,\mathbb{R}^4)$ and where

$$\|\varepsilon_3(\cdot,h,\theta)\|_{\mathbf{L}^\infty(\Omega)} \leqslant C\left(|h|^2 + |\theta|^2\right). \tag{4.5}$$

Thus, by differentiating (4.3) we also deduce that

$$\begin{aligned}
\frac{\partial}{\partial y_j}\left(\nabla Y(\cdot,X)\right)_{kl} &= \Lambda_{4klj}(y)(h,\theta) + \varepsilon_{4klj}(y,h,\theta), && (4.6)\\
\frac{\partial^2 Y_l}{\partial x_j \partial x_k}(X) &= \Lambda_{5klj}(y)(h,\theta) + \varepsilon_{5klj}(y,h,\theta), && (4.7)\\
\frac{\partial^3 Y_l}{\partial x_j \partial x_k \partial x_m}(X) &= \Lambda_{6kljm}(y)(h,\theta) + \varepsilon_{6kljm}(y,h,\theta), && (4.8)
\end{aligned}$$

with the corresponding estimates

$$\|\varepsilon_4(\cdot,h,\theta)\|_{\mathbf{L}^\infty(\Omega)} + \|\varepsilon_5(\cdot,h,\theta)\|_{\mathbf{L}^\infty(\Omega)} + \|\varepsilon_6(\cdot,h,\theta)\|_{\mathbf{L}^\infty(\Omega)} \leqslant C\left(|h|^2 + |\theta|^2\right).$$

Next, from (2.2), $h' = R_\theta \ell$ and $(R_\theta y)' = \omega R_\theta y^\perp$ we deduce

$$\partial_t X(t,y) = \eta(y)\left(R_{\theta(t)}\ell(t) + \omega(t)R_{\theta(t)}y^\perp\right) = \eta(y)\left(\ell(t) + \omega(t)y^\perp\right) + \varepsilon_7(y,\theta,\ell,\omega), \tag{4.9}$$

with

$$\|\varepsilon_7(\cdot,\theta,\ell,\omega)\|_{\mathbf{L}^\infty(\Omega)} \leqslant C\left(|\theta|^2 + |\ell|^2 + |\omega|^2\right).$$

Thus, we obtain from (4.9), (4.4) and (4.5) that

$$\partial_t Y(t,X(t,y)) = -\nabla Y(t,X(t,y))\left(\partial_t X(t,y)\right) = -\eta(y)\left(\ell(t) + \omega(t)y^\perp\right) + \varepsilon_8(h,\theta,\ell,\omega,y), \tag{4.10}$$

with

$$\|\varepsilon_8(h,\theta,\ell,\omega,\cdot)\|_{\mathbf{L}^\infty(\Omega)} \leqslant C\left(|h|^2 + |\theta|^2 + |\ell|^2 + |\omega|^2\right). \tag{4.11}$$

From (4.3), we also deduce that

$$\frac{\partial}{\partial t}\left[\nabla Y(t,X(t,y))\right] = -\left(\ell(t) + \omega(t)y^\perp\right) \otimes \nabla\eta(y) - \eta(y)\omega(t)R_{\pi/2} + \varepsilon_9(y,h,\theta,\ell,\omega), \tag{4.12}$$

with

$$\|\varepsilon_9(\cdot,h,\theta,\ell,\omega)\|_{\mathbf{L}^\infty(\Omega)} \leqslant C\left(|h|^2+|\theta|^2+|\ell|^2+|\omega|^2\right).$$

Finally, we notice that

$$\operatorname{Cof}\begin{bmatrix} a & b \\ c & d\end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a\end{bmatrix}, \tag{4.13}$$

and combining the above relation with (4.4), (4.7), (4.6), and (4.12), we deduce that

$$\begin{aligned}
\operatorname{Cof}(\nabla Y)(t,X(t,y))_{i,j} &= \delta_{i,j} + \Lambda_{10\,ij}(y)(h,\theta) + \varepsilon_{10\,ij}(y,h,\theta), &(4.14)\\
\frac{\partial}{\partial x_\beta}\operatorname{Cof}(\nabla Y)_{ij}(t,X(t,y)) &= \Lambda_{11\,ij\beta}(y)(h,\theta) + \varepsilon_{11\,ij\beta}(y,h,\theta), &(4.15)\\
\frac{\partial^2}{\partial x_\beta^2}\operatorname{Cof}(\nabla Y)_{ij}(t,X(t,y)) &= \Lambda_{12\,ij\beta}(y)(h,\theta) + \varepsilon_{12\,ij\beta}(y,h,\theta), &(4.16)\\
\frac{\partial}{\partial t}\operatorname{Cof}(\nabla Y)_{ij}(t,X(t,y)) &= \Lambda_{13\,ij}(y)(\ell,\omega) + \varepsilon_{13\,ij}(y,h,\theta,\ell,\omega), &(4.17)
\end{aligned}$$

where Λ_{10}, Λ_{11}, Λ_{12}, Λ_{13} are smooth mappings from $\overline{\Omega}$ into $\mathcal{L}(\mathbb{R}^3,\mathbb{R}^k)$ ($k=4$ or $k=8$) and with

$$\|\varepsilon_{10}(\cdot,h,\theta)\|_{\mathbf{L}^\infty(\Omega)} + \|\varepsilon_{11}(\cdot,h,\theta)\|_{\mathbf{L}^\infty(\Omega)} + \|\varepsilon_{12}(\cdot,h,\theta)\|_{\mathbf{L}^\infty(\Omega)} \leqslant C\left(|h|^2+|\theta|^2\right) \tag{4.18}$$

$$\|\varepsilon_{13}(\cdot,h,\theta,\ell,\omega)\|_{\mathbf{L}^\infty(\Omega)} \leqslant C\left(|h|^2+|\theta|^2+|\ell|^2+|\omega|^2\right). \tag{4.19}$$

Proof of Proposition 12. From (2.17), we deduce that

$$\begin{aligned}
[(\mathbf{L}-\Delta)v^S]_i = \sum_{j,k} \frac{\partial^2}{\partial x_j^2}\operatorname{Cof}(\nabla Y)_{ki}(X) v_k^S + 2\sum_{j,k,l}\frac{\partial}{\partial x_j}\operatorname{Cof}(\nabla Y)_{ki}(X)\frac{\partial v_k^S}{\partial y_l}\frac{\partial Y_l}{\partial x_j}(X)\\
+\sum_{j,k,l}\operatorname{Cof}(\nabla Y)_{ki}(X)\frac{\partial v_k^S}{\partial y_l}\frac{\partial^2 Y_l}{\partial x_j^2}(X)\\
+\sum_{j,k,l,m}\left(\operatorname{Cof}(\nabla Y)_{ki}(X)-\delta_{ki}\right)\frac{\partial^2 v_k^S}{\partial y_l\partial y_m}\frac{\partial Y_l}{\partial x_j}(X)\frac{\partial Y_m}{\partial x_j}(X)\\
+\sum_{j,l,m}\frac{\partial^2 v_i^S}{\partial y_l\partial y_m}\left(\frac{\partial Y_l}{\partial x_j}(X)-\delta_{lj}\right)\frac{\partial Y_m}{\partial x_j}(X) + \sum_{j,m}\frac{\partial^2 v_i^S}{\partial y_j\partial y_m}\left(\frac{\partial Y_m}{\partial x_j}(X)-\delta_{mj}\right).
\end{aligned}$$

Combining the above relation with (4.16), (4.15), (4.4), (4.14), (4.7) and the corresponding estimates (4.18), (4.5), (4.8) to bound the terms

$$\frac{\partial^2}{\partial x_j^2}\operatorname{Cof}(\nabla Y)_{ki}(X),\quad \frac{\partial}{\partial x_j}\operatorname{Cof}(\nabla Y)_{ki}(X)\frac{\partial Y_l}{\partial x_j}(X),\quad \operatorname{Cof}(\nabla Y)_{ki}(X)\frac{\partial^2 Y_l}{\partial x_j^2}(X)$$

$$\left(\operatorname{Cof}(\nabla Y)_{ki}(X)-\delta_{ki}\right)\frac{\partial Y_l}{\partial x_j}(X)\frac{\partial Y_m}{\partial x_j}(X),\quad \left(\frac{\partial Y_l}{\partial x_j}(X)-\delta_{lj}\right)\frac{\partial Y_m}{\partial x_j}(X),\quad \frac{\partial Y_m}{\partial x_j}(X)-\delta_{mj},$$

and with the fact that $v^S\in\mathbf{W}^{2,\infty}(\Omega)$, we deduce (2.21) and its corresponding estimate in (2.25).

The proof of (2.23) is similar and uses (4.4) and the fact that $p^S\in W^{1,\infty}(\mathcal{F})$ in the expression

$$[\mathbf{G}p^S - \nabla p^S])_i = \sum_l \frac{\partial p^S}{\partial y_l}\left(\frac{\partial Y_l}{\partial x_i}(X)-\delta_{il}\right).$$

To prove (2.22), we start from the definition (2.16) of the operator $\mathbf{M}$ and we use (4.17), (4.14), (4.10) and the corresponding estimates (4.19), (4.18), (4.11) to bound the terms

$$\frac{\partial}{\partial t}\left[\operatorname{Cof}(\nabla Y)^*\circ X\right],\quad \operatorname{Cof}(\nabla Y)^*\circ X,\quad (\partial_t Y)\circ X.$$

Then with the fact that $v^S\in\mathbf{W}^{1,\infty}(\mathcal{F})$ we deduce (2.22) and (2.26).

Finally, to prove (2.24) as well as its corresponding estimate in (2.25) we use the Taylor's expension

$$\tilde{f}^S(y) = f^S(y + d(h,\theta)) = f^S(y) + \nabla f^S(y) \cdot d(h,\theta) + \int_0^1 (1-s)\nabla^2 f^S(y + sd(h,\theta))d(h,\theta) \cdot d(h,\theta)ds$$

where

$$d(h,\theta) \stackrel{\text{def}}{=} \eta(y)\,[h + (R_\theta - I_2)y] = \eta(y)\left[h + \theta y^\perp\right] + \varepsilon_1(y,\theta).$$

Then we conclude by combining the two above relations and the fact that $f^S \in \mathbf{W}^{2,\infty}(\mathcal{F})$. □

Proof of Proposition 13. By definition (2.18) of the operator $\mathbf{N}$, we easily verify that

$$\begin{aligned}
&[\mathbf{N}(w+v^S)]_i - \left[(v^S\cdot\nabla)v^S + (w\cdot\nabla)v^S + (v^S\cdot\nabla)w + (w\cdot\nabla)w\right]_i \\
&\qquad = \sum_{j,k,r} \operatorname{Cof}(\nabla Y)_{kj}(X)\frac{\partial}{\partial x_j}\operatorname{Cof}(\nabla Y)_{ri}(X)\left(w_k w_r + w_k v_r^S + v_k^S w_r + v_k^S v_r^S\right) \\
&\qquad\qquad + \sum_{k,r}\left(\det((\nabla Y)(X))^2 \frac{\partial X_i}{\partial y_r} - \delta_{ir}\right)\left(v_k^S\frac{\partial v_r^S}{\partial y_k} + w_k\frac{\partial v_r^S}{\partial y_k} + v_k^S\frac{\partial w_r}{\partial y_k} + w_k\frac{\partial w_r}{\partial y_k}\right). \qquad (4.20)
\end{aligned}$$

Using (4.2), (4.4), (4.14), (4.15), we deduce that

$$\sum_j \operatorname{Cof}(\nabla Y)_{kj}(X)\frac{\partial}{\partial x_j}\operatorname{Cof}(\nabla Y)_{ri}(X) = \Lambda_{14\,ikr}(y)(h,\theta) + \varepsilon_{14\,ikr}(y,h,\theta),$$

$$\left(\det((\nabla Y)(X))^2\frac{\partial X_i}{\partial y_r} - \delta_{ir}\right) = \Lambda_{15\,ir}(y)(h,\theta) + \varepsilon_{15\,ir}(y,h,\theta),$$

where Λ_{14}, Λ_{15} are smooth mappings from $\overline{\Omega}$ into $\mathcal{L}(\mathbb{R}^3,\mathbb{R}^k)$ ($k=4$ or $k=8$) and where

$$\|\varepsilon_{14}(\cdot,h,\theta)\|_{\mathbf{L}^\infty(\Omega)} + \|\varepsilon_{15}(\cdot,h,\theta)\|_{\mathbf{L}^\infty(\Omega)} \leqslant C\left(|h|^2 + |\theta|^2\right).$$

Combining the above relations with (4.20), we deduce (2.27). □

4.2 Estimates on L, M, G and N

In order to prove Proposition 27, we first prove some estimates for $\mathbf{L}$, $\mathbf{M}$, $\mathbf{G}$ and $\mathbf{N}$ that are involved in the definition (2.40) of $F(\mathbf{X},q)$.

Lemma 31. *There exists $C>0$ such that for all $\mathbf{F}\in B_{\mathcal{E}}(0,\delta)$, the solution $(w,q,\ell,\omega,h,\theta)$ of* (3.2)–(3.9), (3.10) *and* (3.30) *associated with* $\mathbf{F}$ *satisfies*

$$\|[\mathbf{L}-\Delta]w\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F}))} \leqslant C(\delta + \|[w^0,\ell^0,\omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3})^2, \qquad (4.21)$$

$$\|[\mathbf{K}-\mathrm{Id}]\partial_t w\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F}))} \leqslant C(\delta + \|[w^0,\ell^0,\omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3})^2, \qquad (4.22)$$

$$\|[\mathbf{G}-\nabla]q\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F}))} \leqslant C(\delta + \|[w^0,\ell^0,\omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3})^2, \qquad (4.23)$$

$$\|[\mathbf{M}w]\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F}))} \leqslant C(\delta + \|[w^0,\ell^0,\omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3})^2, \qquad (4.24)$$

Lemma 32. *There exists $C>0$ such that for all $\mathbf{F}\in B_{\mathcal{E}}(0,\delta)$, the solution $(w,q,\ell,\omega,h,\theta)$ of* (3.2)–(3.9), (3.10) *and* (3.30) *associated with* $\mathbf{F}$ *satisfies*

$$\left\|P[\mathbf{N}^S(w),0,0,0,0]^*\right\|_{L^2_\sigma(\mathcal{V}')} + \left\|\chi[\mathbf{N}^S(w),0,0,0,0]^*\right\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^6)} \leqslant C(\delta + \|[w^0,\ell^0,\omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3})^2.$$

Gathering Lemma 31, Lemma 32, Proposition 12, and Proposition 13, one can prove Proposition 27. The details of such a proof are omitted but can be obtained by an easy and tedious calculation.

Proof of Lemma 31. The proof is similar to the proof of Proposition 12.

From the estimates (4.4), (4.7), (4.14), (4.15), and (4.16), we deduce

$$\left\|\frac{\partial Y_l}{\partial x_j}(X)-\delta_{lj}\right\|_{L^\infty(\Omega)}+\left\|\frac{\partial^2 Y_l}{\partial x_j\partial x_k}(X)\right\|_{L^\infty(\Omega)}+\|\mathrm{Cof}(\nabla Y)_{ki}(X)-\delta_{ki}\|_{L^\infty(\Omega)}$$
$$+\left\|\frac{\partial}{\partial x_j}\mathrm{Cof}(\nabla Y)_{ki}(X)\right\|_{L^\infty(\Omega)}+\left\|\frac{\partial^2}{\partial x_j^2}\mathrm{Cof}(\nabla Y)_{ki}(X)\right\|_{L^\infty(\Omega)}\leqslant C\left(|h(t)|+|\theta(t)|\right).$$

From the definition of $\mathbf{L}$, we deduce from the above relation that

$$\|[\mathbf{L}-\Delta]w(t)\|_{\mathbf{L}^2(\mathcal{F})}\leqslant C\left(|h(t)|+|\theta(t)|\right)\|w(t)\|_{\mathbf{H}^2(\mathcal{F})}. \tag{4.25}$$

On the other hand, the fact that $\mathbf{F}\in B_{\mathcal{E}}(0,\delta)$ with (3.43) in Corollary 25 yields

$$|h(t)|+|\theta(t)|\leqslant C\chi(t)(\delta+\|[w^0,\ell^0,\omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3}).$$

Moreover, from $\mathbf{F}\in B_{\mathcal{E}}(0,\delta)$ with (3.42) in Corollary 25 we deduce that

$$\|\chi w\|_{L^2_\sigma(\mathbf{H}^2(\mathcal{F}))}\leqslant C(\delta+\|[w^0,\ell^0,\omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3}).$$

Combining the above estimates with (4.25), we deduce (4.21). Estimates (4.22), (4.23) and (4.24) are done in a similar way. □

Proof of Lemma 32. From (2.28), we have

$$[\mathbf{N}^S(w)]_i\overset{\text{def}}{=}[(w\cdot\nabla)w]_i+\sum_{j,k,r}\mathrm{Cof}(\nabla Y)_{kj}(X)\frac{\partial}{\partial x_j}\mathrm{Cof}(\nabla Y)_{ri}(X)\left(w_kw_r+w_kv_r^S+v_k^Sw_r\right)$$
$$+\sum_{k,r}\left(\det((\nabla Y)(X))^2\frac{\partial X_i}{\partial y_r}-\delta_{ir}\right)\left(w_k\frac{\partial v_r^S}{\partial y_l}+v_k^S\frac{\partial w_r}{\partial y_l}+w_k\frac{\partial w_r}{\partial y_l}\right).$$

We decompose $\mathbf{N}^S$ as follows:

$$\mathbf{N}^S(w)=\bar{\mathbf{N}}(w,\ell,\omega)+\hat{\mathbf{N}}(w,\ell,\omega,h,\theta). \tag{4.26}$$

with

$$\bar{\mathbf{N}}(w,\ell,\omega)\overset{\text{def}}{=}((w-\ell-\omega y^\perp)\cdot\nabla)w. \tag{4.27}$$

The estimate of $\hat{\mathbf{N}}$ can be done as in the proof of Lemma 31. Then we only give details on the estimate of $\bar{\mathbf{N}}$. From the Hölder's inequality and the Sobolev embedding theorem, we deduce that

$$\chi\left\|w_k\frac{\partial w_r}{\partial y_l}\right\|_{L^2(\mathcal{F})}\leqslant C\chi\|w\|_{\mathbf{H}^{1/2}(\mathcal{F})}\|w\|_{\mathbf{H}^{3/2}(\mathcal{F})}.$$

The above estimate and the interpolation inequalities

$$\begin{aligned}\chi^{1/2}\|w\|_{\mathbf{H}^{1/2}(\mathcal{F})}&\leqslant C\left(\chi\|w\|_{\mathbf{H}^1(\mathcal{F})}\right)^{1/2}\left(\|w\|_{\mathbf{L}^2(\mathcal{F})}\right)^{1/2}\\ \chi^{1/2}\|w\|_{\mathbf{H}^{3/2}(\mathcal{F})}&\leqslant C\left(\chi\|w\|_{\mathbf{H}^2(\mathcal{F})}\right)^{1/2}\left(\|w\|_{\mathbf{H}^1(\mathcal{F})}\right)^{1/2}\end{aligned}$$

yield

$$\left\|\chi w_k\frac{\partial w_r}{\partial y_l}\right\|_{L^2_\sigma(L^2(\mathcal{F}))}\leqslant C\left(\|\chi w\|_{L^\infty_\sigma(\mathbf{H}^1(\mathcal{F}))}+\|w\|_{L^\infty_\sigma(\mathbf{L}^2(\mathcal{F}))}\right)\left(\|\chi w\|_{L^2_\sigma(\mathbf{H}^2(\mathcal{F}))}+\|w\|_{L^2_\sigma(\mathbf{H}^1(\mathcal{F}))}\right). \tag{4.28}$$

We note that

$$\left\|((\ell+\omega y^\perp)\cdot\nabla)w\right\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F}))}\leqslant C\|w\|_{L^2_\sigma(\mathbf{H}^1(\mathcal{F}))}\left(\|\ell\|_{L^\infty_\sigma(\mathbb{R}^2)}+\|\omega\|_{L^\infty_\sigma(\mathbb{R})}\right).$$

The above estimate and (4.28) yield

$$\|\chi\bar{\mathbf{N}}(w,\ell,\omega)\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F}))}\leqslant C\left(\|\chi w\|_{L^\infty_\sigma(\mathbf{H}^1(\mathcal{F}))}+\|(w,\ell,\omega,0,0)\|_{L^\infty_\sigma(\mathcal{H})}\right)\left(\|\chi w\|_{L^2_\sigma(\mathbf{H}^2(\mathcal{F}))}+\|w\|_{L^2_\sigma(\mathbf{H}^1(\mathcal{F}))}\right). \tag{4.29}$$

To estimate $\langle P\left[\bar{\mathbf{N}}(w,\ell,\omega),0,0,0,0\right]^*$ in $L^2_\sigma(\mathcal{V}')$, we notice that it is defined by duality as

$$\langle P\left[\bar{\mathbf{N}}(w,\ell,\omega),0,0,0,0\right]^*,\left[\varphi,\xi,\zeta,a,b\right]^*\rangle \stackrel{\text{def}}{=} -\int_{\mathcal{F}}\Big((w-\ell-\omega y^\perp)\cdot\nabla\Big)\varphi\cdot w\,dx$$

and thus

$$\left\|\langle P\left[\bar{\mathbf{N}}(w,\ell,\omega),0,0,0,0\right]^*\right\|_{L^2_\sigma(\mathcal{V}')} \leqslant C\left(\|w\|_{L^2_\sigma(\mathbf{H}^1(\mathcal{F}))}+\|\ell\|_{L^\infty_\sigma(\mathbb{R}^2)}+\|\omega\|_{L^\infty_\sigma(\mathbb{R})}\right)^2. \tag{4.30}$$

Finally, we conclude by remarking that $\mathbf{F}\in B_{\mathcal{E}}(0,\delta)$ with (3.42) in Corollary 25 implies that the right hand sides of equalities (4.29), (4.30) are bounded by $C(\delta+\|[w^0,\ell^0,\omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3})^2$. □

4.3 Estimates of the differences

Assume $\mathbf{F}^{(1)},\mathbf{F}^{(2)}\in B_{\mathcal{E}}(0,\delta)$, with δ small enough. The solutions $(w^{(i)},q^{(i)},\ell^{(i)},\omega^{(i)},h^{(i)},\theta^{(i)})$ of (3.2)–(3.9), (3.10) and (3.30) associated with $\mathbf{F}^{(i)}$ for $i=1,2$ satisfy the results of Corollary 25 and in particular,

$$\begin{aligned}&\|w^{(i)}\|_{L^2_\sigma(\mathbf{H}^1(\mathcal{F}))}+\|w^{(i)}\|_{L^\infty_\sigma(\mathbf{L}^2(\mathcal{F}))}+\|(\ell^{(i)},\omega^{(i)},h^{(i)},\theta^{(i)})\|_{(L^\infty_\sigma)^6}+\|\chi w^{(i)}\|_{W_\sigma(\mathbf{H}^2(\mathcal{F}),\mathbf{L}^2(\mathcal{F}))}\\&+\|\chi(\ell^{(i)},\omega^{(i)},h^{(i)},\theta^{(i)})\|_{(H^1_\sigma)^6}+\|\chi q^{(i)}\|_{L^2_\sigma(H^1(\mathcal{F}))}\leqslant C(\delta+\|[w^0,\ell^0,\omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3}),\end{aligned} \tag{4.31}$$

and

$$|h^{(i)}(t)|+|\theta^{(i)}(t)|\leqslant\chi(t)C(\delta+\|[w^0,\ell^0,\omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3}, \tag{4.32}$$

for $i=1,2$.

Moreover, we notice that

$$(w,q,\ell,\omega,h,\theta)\stackrel{\text{def}}{=}(w^{(1)},q^{(1)},\ell^{(1)},\omega^{(1)},h^{(1)},\theta^{(1)})-(w^{(2)},q^{(2)},\ell^{(2)},\omega^{(2)},h^{(2)},\theta^{(2)})$$

is also solution of the system (3.2)–(3.9), (3.10) and (3.30) associated with a zero initial condition and with $\mathbf{F}\stackrel{\text{def}}{=}\mathbf{F}^{(1)}-\mathbf{F}^{(2)}$. In particular, using again Corollary 25,

$$\begin{aligned}\|w\|_{L^2_\sigma(\mathbf{H}^1(\mathcal{F}))}+&\|w\|_{L^\infty_\sigma(\mathbf{L}^2(\mathcal{F}))}+\|(\ell,\omega,h,\theta)\|_{(L^\infty_\sigma)^6}+\|\chi w\|_{W_\sigma(\mathbf{H}^2(\mathcal{F}),\mathbf{L}^2(\mathcal{F}))}+\|\chi(\ell,\omega,h,\theta)\|_{(H^1_\sigma)^6}\\&+\|\chi q\|_{L^2_\sigma(H^1(\mathcal{F}))}\leqslant C\|\mathbf{F}\|_{\mathcal{E}},\end{aligned} \tag{4.33}$$

and

$$|h(t)|+|\theta(t)|\leqslant\chi(t)C\|\mathbf{F}\|_{\mathcal{E}}.$$

In order to prove Proposition 28, we need to estimate the difference

$$\begin{aligned}&F(\mathbf{X}^{(1)},q^{(1)})-F(\mathbf{X}^{(2)},q^{(2)})\\&=[(\mathrm{Id}-\mathbf{K}^{(1)})\partial_t(w^{(1)}-w^{(2)})]+\nu[(\mathbf{L}^{(1)}-\Delta)(w^{(1)}-w^{(2)})]-[\mathbf{M}^{(1)}(w^{(1)}-w^{(2)})]+[(\nabla-\mathbf{G}^{(1)})(q^{(1)}-q^{(2)})]\\&\quad+[(\mathbf{K}^{(2)}-\mathbf{K}^{(1)})\partial_t w^{(2)}]+\nu[(\mathbf{L}^{(1)}-\mathbf{L}^{(2)})w^{(2)}]-[(\mathbf{M}^{(1)}-\mathbf{M}^{(2)})w^{(2)}]+[(\mathbf{G}^{(2)}-\mathbf{G}^{(1)})q^{(2)}]\\&\quad-\varepsilon_L(h^{(1)},\theta^{(1)})+\varepsilon_L(h^{(2)},\theta^{(2)})-\varepsilon_M(h^{(1)},\theta^{(1)},\ell^{(1)},\omega^{(1)})+\varepsilon_M(h^{(2)},\theta^{(2)},\ell^{(2)},\omega^{(2)})\\&\quad-\varepsilon_N(h^{(1)},\theta^{(1)})+\varepsilon_N(h^{(2)},\theta^{(2)})-\varepsilon_G(h^{(1)},\theta^{(1)})+\varepsilon_G(h^{(2)},\theta^{(2)})+\varepsilon_f(h^{(1)},\theta^{(1)})-\varepsilon_f(h^{(2)},\theta^{(2)})\\&\quad-[(\mathbf{N}^S)^{(1)}(w^{(1)})]+[(\mathbf{N}^S)^{(2)}(w^{(2)})].\end{aligned} \tag{4.34}$$

The first 4 terms of the above right hand side can be estimated by using a proof similar to the proof of Lemma 31. More precisely, using (4.33), (4.31) and (4.32), we obtain

$$\begin{aligned}\Big\|&[(\mathrm{Id}-\mathbf{K}^{(1)})\partial_t(w^{(1)}-w^{(2)})]+\nu[(\mathbf{L}^{(1)}-\Delta)(w^{(1)}-w^{(2)})]-[\mathbf{M}^{(1)}(w^{(1)}-w^{(2)})]\\&+[(\nabla-\mathbf{G}^{(1)})(q^{(1)}-q^{(2)})]\Big\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F}))}\leqslant C(\delta+\|[w^0,\ell^0,\omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3})\|\mathbf{F}\|_{\mathcal{E}}.\end{aligned}$$

For the other terms in (4.34), we need the following results:

Lemma 33. *There exists a positive constant C such that for all $\mathbf{F}^{(1)}, \mathbf{F}^{(2)} \in B_{\mathcal{E}}(0,\delta)$, the solutions $(w^{(i)}, q^{(i)}, \ell^{(i)}, \omega^{(i)}, h^{(i)}, \theta^{(i)})$ of (3.2)–(3.9), (3.10) and (3.30) associated with $\mathbf{F}^{(i)}$ for $i = 1, 2$ satisfy*

$$\begin{aligned}
\left\|[(\mathbf{L}^{(1)} - \mathbf{L}^{(2)})w^{(2)}]\right\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F}))} &\leqslant C(\delta + \|[w^0, \ell^0, \omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3})\|\mathbf{F}\|_{\mathcal{E}},\\
\left\|[(\mathbf{K}^{(1)} - \mathbf{K}^{(2)})\partial_t w^{(2)}]\right\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F}))} &\leqslant C(\delta + \|[w^0, \ell^0, \omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3})\|\mathbf{F}\|_{\mathcal{E}},\\
\left\|[(\mathbf{G}^{(1)} - \mathbf{G}^{(2)})q^{(2)}]\right\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F}))} &\leqslant C(\delta + \|[w^0, \ell^0, \omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3})\|\mathbf{F}\|_{\mathcal{E}},\\
\left\|[(\mathbf{M}^{(1)} - \mathbf{M}^{(2)})w^{(2)}]\right\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F}))} &\leqslant C(\delta + \|[w^0, \ell^0, \omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3})\|\mathbf{F}\|_{\mathcal{E}}.
\end{aligned}$$

Lemma 34. *There exists a positive constant C such that for all $\mathbf{F}^{(1)}, \mathbf{F}^{(2)} \in B_{\mathcal{E}}(0,\delta)$, the solutions $(w^{(i)}, q^{(i)}, \ell^{(i)}, \omega^{(i)}, h^{(i)}, \theta^{(i)})$ of (3.2)–(3.9), (3.10) and (3.30) associated with $\mathbf{F}^{(i)}$ for $i = 1, 2$ satisfy*

$$\begin{aligned}
\left\|P[\mathbf{N}^{S\,(1)}(w^{(1)}) - \mathbf{N}^{S\,(2)}(w^{(2)}), 0,0,0,0]^*\right\|_{L^2_\sigma(\mathcal{V}')} + \left\|\chi[\mathbf{N}^{S\,(1)}(w^{(1)}) - \mathbf{N}^{S\,(2)}(w^{(2)}), 0,0,0,0]^*\right\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^6)}&\\
\leqslant C(\delta + \|[w^0, \ell^0, \omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3})\|\mathbf{F}\|_{\mathcal{E}}.&
\end{aligned}$$

Proposition 35. *There exists a positive constant C such that for all $\mathbf{F}^{(1)}, \mathbf{F}^{(2)} \in B_{\mathcal{E}}(0,\delta)$, the solutions $(w^{(i)}, q^{(i)}, \ell^{(i)}, \omega^{(i)}, h^{(i)}, \theta^{(i)})$ of (3.2)–(3.9), (3.10) and (3.30) associated with $\mathbf{F}^{(i)}$ for $i = 1, 2$ satisfy*

$$\begin{aligned}
&\left\|\varepsilon_L(\cdot, h^{(1)}, \theta^{(1)}) - \varepsilon_L(\cdot, h^{(2)}, \theta^{(2)})\right\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F}))} + \left\|\varepsilon_G(\cdot, h^{(1)}, \theta^{(1)}) - \varepsilon_G(\cdot, h^{(2)}, \theta^{(2)})\right\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F}))}\\
&+ \left\|\varepsilon_N(\cdot, h^{(1)}, \theta^{(1)}) - \varepsilon_N(\cdot, h^{(2)}, \theta^{(2)})\right\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F}))} + \left\|\varepsilon_f(\cdot, h^{(1)}, \theta^{(1)}) - \varepsilon_f(\cdot, h^{(2)}, \theta^{(2)})\right\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F}))}\\
&+ \left\|\varepsilon_M(\cdot, h^{(1)}, \theta^{(1)}, \ell^{(1)}, \omega^{(1)}) - \varepsilon_M(\cdot, h^{(2)}, \theta^{(2)}, \ell^{(2)}, \omega^{(2)})\right\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F}))}\\
&\leqslant C(\delta + \|[w^0, \ell^0, \omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3})\|\mathbf{F}\|_{\mathcal{E}}.
\end{aligned}$$

Combining the above results, one can prove Proposition 28. Let us prove now Lemma 33, Lemma 34, and Proposition 35.

Proof of Lemma 33. First, using the definition (2.2) of the change of variables, we obtain

$$\|X(\cdot, h^{(1)}, \theta^{(1)}) - X(\cdot, h^{(2)}, \theta^{(2)})\|_{\mathbf{W}^{2,\infty}(\Omega)} \leqslant C(|h| + |\theta|).$$

Then, using the classical formula

$$\nabla Y^{(1)}(X^{(1)}) - \nabla Y^{(2)}(X^{(2)}) = \nabla Y^{(2)}(X^{(2)}) \left(\nabla X^{(2)} - \nabla X^{(1)}\right) \nabla Y^{(1)}(X^{(1)}), \tag{4.35}$$

we deduce

$$\begin{aligned}
\|\nabla Y^{(1)}(X^{(1)}) - \nabla Y^{(2)}(X^{(2)})\|_{\mathbf{L}^\infty(\Omega)} + \left\|\frac{\partial^2 Y_l^{(1)}}{\partial x_j \partial x_k}(X^{(1)}) - \frac{\partial^2 Y_l^{(2)}}{\partial x_j \partial x_k}(X^{(2)})\right\|_{\mathbf{L}^\infty(\Omega)}&\\
+ \left\|\frac{\partial^3 Y_l^{(1)}}{\partial x_j \partial x_k \partial x_m}(X^{(1)}) - \frac{\partial^3 Y_l^{(2)}}{\partial x_j \partial x_k \partial x_m}(X^{(2)})\right\|_{\mathbf{L}^\infty(\Omega)} &\leqslant C(|h| + |\theta|).
\end{aligned}$$

By differentiating (2.2) with respect to time, we also derive

$$\|\partial_t X(\cdot, h^{(1)}, \theta^{(1)}) - \partial_t X(\cdot, h^{(2)}, \theta^{(2)})\|_{\mathbf{L}^\infty(\Omega)} \leqslant C(|(h^{(1)})' - (h^{(2)})'| + |(\theta^{(1)})' - (\theta^{(2)})'|)$$

that yields

$$\begin{aligned}
&\|\partial_t X(\cdot, h^{(1)}, \theta^{(1)}) - \partial_t X(\cdot, h^{(2)}, \theta^{(2)})\|_{L^\infty(\mathbf{L}^\infty(\Omega))}\\
&\qquad\leqslant C(\|\ell\|_{(L^\infty)^2} + \|\omega\|_{L^\infty} + \|h_F^{(1)} - h_F^{(2)}\|_{(L^\infty)^2} + \|\theta_F^{(1)} - \theta_F^{(2)}\|_{L^\infty}).
\end{aligned}$$

The above result and the first relation of (4.10) yield

$$\begin{aligned}&\|\partial_t Y^{(1)}(\cdot,X^{(1)})-\partial_t Y^{(2)}(\cdot,X^{(2)})\|_{L^\infty(\mathbf{L}^\infty(\Omega))}\\&\qquad\leqslant C(\|\ell\|_{(L^\infty)^2}+\|\omega\|_{L^\infty}+\|h\|_{(L^\infty)^2}+\|\theta\|_{L^\infty}+\|h_F^{(1)}-h_F^{(2)}\|_{(L^\infty)^2}+\|\theta_F^{(1)}-\theta_F^{(2)}\|_{L^\infty}).\end{aligned}$$

Using againg (4.35), we deduce

$$\begin{aligned}&\|\partial_t \nabla Y^{(1)}(\cdot,X^{(1)})-\partial_t \nabla Y^{(2)}(\cdot,X^{(2)})\|_{L^\infty(\mathbf{L}^\infty(\Omega))}\\&\qquad\leqslant C(\|\ell\|_{(L^\infty)^2}+\|\omega\|_{L^\infty}+\|h\|_{(L^\infty)^2}+\|\theta\|_{L^\infty}+\|h_F^{(1)}-h_F^{(2)}\|_{(L^\infty)^2}+\|\theta_F^{(1)}-\theta_F^{(2)}\|_{L^\infty}).\end{aligned}$$

Finally, using (4.13) and (4.33), we deduce

$$\begin{aligned}&\left\|\operatorname{Cof}(\nabla Y^{(1)})(X^{(1)})-\operatorname{Cof}(\nabla Y^{(2)})(X^{(2)})\right\|_{L^\infty(\mathbf{W}^{2,\infty}(\Omega))}\\&\qquad+\left\|\frac{\partial}{\partial t}\operatorname{Cof}(\nabla Y^{(1)})(X^{(1)})-\frac{\partial}{\partial t}\operatorname{Cof}(\nabla Y^{(2)})(X^{(2)})\right\|_{L^\infty(\mathbf{L}^\infty(\Omega))}\leqslant C\|\mathbf{F}\|_{\mathcal{E}}.\end{aligned}$$

The proof concludes by combining the above results with (2.15), (2.16), (2.17), (2.19) and with (4.31). $\square$

Proof of Lemma 34. As in the proof of Lemma 32, we use the decomposition (4.26), (4.27) of $\mathbf{N}$:

$$\begin{aligned}\mathbf{N}^{S(1)}(w^{(1)})-\mathbf{N}^{S(2)}(w^{(2)})&=\bar{\mathbf{N}}^{(1)}(w^{(1)},\ell^{(1)},\omega^{(1)})-\bar{\mathbf{N}}^{(2)}(w^{(2)},\ell^{(2)},\omega^{(2)})\\&\quad+\hat{\mathbf{N}}^{(1)}(w^{(1)},\ell^{(1)},\omega^{(1)},h^{(1)},\theta^{(1)})-\hat{\mathbf{N}}^{(2)}(w^{(2)},\ell^{(2)},\omega^{(2)},h^{(2)},\theta^{(2)}).\end{aligned}$$

The difference

$$\hat{\mathbf{N}}^{(1)}(w^{(1)},\ell^{(1)},\omega^{(1)},h^{(1)},\theta^{(1)})-\hat{\mathbf{N}}^{(2)}(w^{(2)},\ell^{(2)},\omega^{(2)},h^{(2)},\theta^{(2)})$$

can be estimated in $L^2_\sigma(\mathbf{L}^2(\mathcal{F}))$ by using the estimates of the coefficients derived in the proof of Lemma 33. For the other term, we notice that

$$\bar{\mathbf{N}}^{(1)}(w^{(1)},\ell^{(1)},\omega^{(1)})-\bar{\mathbf{N}}^{(2)}(w^{(2)},\ell^{(2)},\omega^{(2)})=((w^{(1)}-\ell^{(1)}-\omega^{(1)}y^\perp)\cdot\nabla)w+((w-\ell-\omega y^\perp)\cdot\nabla)w^{(2)}.$$

Then, following the proof of Lemma 32 and using (4.31), (4.33), we can prove

$$\left\|\chi\left(\bar{\mathbf{N}}^{(1)}(w^{(1)},\ell^{(1)},\omega^{(1)})-\bar{\mathbf{N}}^{(2)}(w^{(2)},\ell^{(2)},\omega^{(2)})\right)\right\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F}))}\leqslant C(\delta+\|[w^0,\ell^0,\omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3})\|\mathbf{F}\|_{\mathcal{E}}.$$

$$\left\|\langle P\left[\bar{\mathbf{N}}^{(1)}(w^{(1)},\ell^{(1)},\omega^{(1)})-\bar{\mathbf{N}}^{(2)}(w^{(2)},\ell^{(2)},\omega^{(2)}),0,0,0,0\right]^*\right\|_{L^2_\sigma(\mathcal{V}')}\leqslant C(\delta+\|[w^0,\ell^0,\omega^0]^*\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^3})\|\mathbf{F}\|_{\mathcal{E}}.$$

$\square$

Proof of Proposition 35. First we notice that

$$\left|\left(R_{\theta^{(1)}}-I_2-\theta^{(1)}R_{\pi/2}\right)-\left(R_{\theta^{(2)}}-I_2-\theta^{(2)}R_{\pi/2}\right)\right|\leqslant C\left(|\theta^{(1)}|+|\theta^{(2)}|\right)|\theta|,$$

and we deduce from the above estimate and from (4.1), (4.2) that

$$\|\varepsilon_1(\cdot,\theta^{(1)})-\varepsilon_1(\cdot,\theta^{(2)})\|_{\mathbf{L}^\infty(\Omega)}+\|\varepsilon_2(\cdot,\theta^{(1)})-\varepsilon_2(\cdot,\theta^{(2)})\|_{(\mathbf{L}^\infty(\Omega))^4}\leqslant C\left(|\theta^{(1)}|+|\theta^{(2)}|\right)|\theta|. \tag{4.36}$$

To obtain the estimates on ε_3 in (4.4), we use the classical formula (4.35) and we combine it

$$\nabla X^{(i)}=-\Lambda_3(h^{(i)},\theta^{(i)})+\varepsilon_2(h^{(i)},\theta^{(i)}),$$

we deduce that

$$\begin{aligned}\varepsilon_3(\cdot,h^{(1)},\theta^{(1)})-\varepsilon_3(\cdot,h^{(2)},\theta^{(2)})&=\left(\nabla Y^{(2)}(X^{(2)})-I_2\right)\left(\Lambda_3(h^{(1)},\theta^{(1)})-\Lambda_3(h^{(2)},\theta^{(2)})\right)\\&\quad+\nabla Y^{(2)}(X^{(2)})\left(\Lambda_3(h^{(1)},\theta^{(1)})-\Lambda_3(h^{(2)},\theta^{(2)})\right)\left(\nabla Y^{(1)}(X^{(1)})-I_2\right)\\&\quad+\nabla Y^{(2)}(X^{(2)})\left(\varepsilon_2(\cdot,h^{(1)},\theta^{(1)})-\varepsilon_2(\cdot,h^{(2)},\theta^{(2)})\right)\nabla Y^{(1)}(X^{(1)}).\end{aligned}$$

The above relation, (4.4) and (4.36) yield

$$\|\varepsilon_3(\cdot,h^{(1)},\theta^{(1)})-\varepsilon_3(\cdot,h^{(2)},\theta^{(2)})\|_{\mathbf{L}^\infty(\Omega)} \leqslant C\left(|h^{(1)}|+|h^{(2)}|+|\theta^{(1)}|+|\theta^{(2)}|\right)(|h|+|\theta|)\,.$$

Differentiating relation (4.35), we deduce in a similar way that

$$\begin{aligned}&\|\varepsilon_5(\cdot,h^{(1)},\theta^{(1)})-\varepsilon_5(\cdot,h^{(2)},\theta^{(2)})\|_{\mathbf{L}^\infty(\Omega)}+\|\varepsilon_6(\cdot,h^{(1)},\theta^{(1)})-\varepsilon_6(\cdot,h^{(2)},\theta^{(2)})\|_{\mathbf{L}^\infty(\Omega)}\\&+\|\varepsilon_{10}(\cdot,\theta^{(1)})-\varepsilon_{10}(\cdot,\theta^{(2)})\|_{\mathbf{L}^\infty(\Omega)}+\|\varepsilon_{11}(\cdot,\theta^{(1)})-\varepsilon_{11}(\cdot,\theta^{(2)})\|_{\mathbf{L}^\infty(\Omega)}+\|\varepsilon_{12}(\cdot,\theta^{(1)})-\varepsilon_{12}(\cdot,\theta^{(2)})\|_{\mathbf{L}^\infty(\Omega)}\\&\leqslant C\left(|h^{(1)}|+|h^{(2)}|+|\theta^{(1)}|+|\theta^{(2)}|\right)(|h|+|\theta|)\,,\end{aligned}\tag{4.37}$$

and that

$$\begin{aligned}&\|\varepsilon_9(\cdot,h^{(1)},\theta^{(1)},\ell^{(1)},\omega^{(1)})-\varepsilon_9(\cdot,h^{(2)},\theta^{(2)},\ell^{(2)},\omega^{(2)})\|_{\mathbf{L}^\infty(\Omega)}\\&+\|\varepsilon_{13}(\cdot,h^{(1)},\theta^{(1)},\ell^{(1)},\omega^{(1)})-\varepsilon_{13}(\cdot,h^{(2)},\theta^{(2)},\ell^{(2)},\omega^{(2)})\|_{\mathbf{L}^\infty(\Omega)}\\&\leqslant C\left(|h^{(1)}|+|h^{(2)}|+|\theta^{(1)}|+|\theta^{(2)}|+|\ell^{(1)}|+|\ell^{(2)}|+|\omega^{(1)}|+|\omega^{(2)}|\right)(|h|+|\theta|+|\ell|+|\omega|)\,.\end{aligned}\tag{4.38}$$

Thus, from (4.9) and from the easily deduced explicit expression of ε_8 we obtain

$$\|\partial_t X^{(2)}-\partial_t X^{(1)}-\eta(y)\ell+\omega y^\perp)\|_{\mathbf{L}^\infty(\Omega)} \leqslant C|\theta|(|\ell^{(2)}|+|\omega^{(1)}|)+|\theta^{(1)}||\ell|+|\omega||\theta^{(2)}|.\tag{4.39}$$

On the other hand, with (4.10) we obtain

$$\begin{aligned}\varepsilon_8(\cdot,h^{(1)},\theta^{(1)},\ell^{(1)},\omega^{(1)})-\varepsilon_8(\cdot,h^{(2)},\theta^{(2)},\ell^{(2)},\omega^{(2)})&=\eta(y)(\ell+\omega y^\perp)+\partial_t X^{(1)}-\partial_t X^{(2)}\\+(\nabla Y^{(1)}(X^{(1)})-\nabla Y^{(2)}(X^{(2)}))\partial_t X^{(1)}&+(\nabla Y^{(2)}(X^{(2)})-\mathrm{Id})(\partial_t X^{(1)}-\partial_t X^{(2)}).\end{aligned}$$

Then from (4.9), (4.39), (4.4), (4.36) we deduce that

$$\begin{aligned}&\|\varepsilon_8(\cdot,h^{(1)},\theta^{(1)},\ell^{(1)},\omega^{(1)})-\varepsilon_8(\cdot,h^{(2)},\theta^{(2)},\ell^{(2)},\omega^{(2)})\|_{\mathbf{L}^\infty(\Omega)}\\&\leqslant C(|h^{(1)}|+|h^{(2)}|+|\theta^{(1)}|+|\theta^{(2)}|+|\ell^{(1)}|+|\ell^{(2)}|+|\omega^{(1)}|+|\omega^{(2)}|)(|h|+|\theta|+|\ell|+|\omega|)\,.\end{aligned}\tag{4.40}$$

The estimate for ε_L, ε_G, ε_M and ε_N is a direct consequence of estimates (4.36),(4.37), (4.38) and of (4.40).

It remains to obtain the bound for ε_f. To do this, we notice that by definition,

$$\varepsilon_f^{(1)}-\varepsilon_f^{(2)}=f^S(X^{(1)})-f^S(X^{(2)})-\eta\nabla f^S(y)\cdot(h+\theta y^\perp).$$

Using Taylor's theorem and recalling that $X^{(1)}-X^{(2)}=h+\theta y^\perp+\varepsilon_1(\theta,y)$, we deduce that in (t,y):

$$\begin{aligned}\varepsilon_f^{(1)}-\varepsilon_f^{(2)} &= \nabla f^S(X^{(2)})\cdot(X^{(1)}-X^{(2)})-\eta\nabla f^S\cdot(h+\theta y^\perp)\\&\quad+\int_0^1(1-s)\nabla^2 f^S((1-s)X^{(1)}+sX^{(2)})(X^{(1)}-X^{(2)})\cdot(X^{(1)}-X^{(2)})ds,\\&=\eta(\nabla f^S(X^{(2)})-\nabla f^S)\cdot(h+\theta y^\perp)+\nabla f^S(X^{(2)})\cdot\varepsilon_1(\theta,y)\\&\quad+\int_0^1(1-s)\nabla^2 f^S((1-s)X^{(1)}+sX^{(2)})(X^{(1)}-X^{(2)})\cdot(X^{(1)}-X^{(2)})ds.\end{aligned}$$

where we have written

$$X^{(i)}=X(t,y,h^{(i)},\theta^{(i)}),\quad \varepsilon_f^{(i)}=\varepsilon_f(t,y,h^{(i)},\theta^{(i)}).$$

Finally, by using that $f^S\in\mathbf{W}^{2,\infty}(\mathcal{F})$ we deduce the result. □

5 The three-dimensional case

We sketch in this section the proof of Theorem 6. It is similar to the proof of Theorem 2 and we only point out the main differences.

As in the 2D case, one has to perform a change of variables to write the system in the fixed domain $\mathcal{F}(h^S, R^S)$. We use the same kind of change of variables than in the 2D case, more precisely we replace (2.2) by

$$X(t,y) \stackrel{\text{def}}{=} y + \eta(y)\left[h(t) + R(t)(R^S)^*(y - h^S) - y\right].$$

Then we consider the transformation (2.5) combined with the 3D version of (2.6)

$$\ell(t) \stackrel{\text{def}}{=} R^S(R(t))^* V(t), \quad \omega(t) \stackrel{\text{def}}{=} R^S(R(t))^* r(t)$$

and we also set

$$Q(t) \stackrel{\text{def}}{=} R(t)(R^S)^* - \mathrm{Id} \quad H(t) \stackrel{\text{def}}{=} h(t) - h^S.$$

The system satisfied by $w = \tilde{v} - v^S$, $q = \tilde{p} - p^S$, ℓ, ω, H and Q can be written as

$$\partial_t w - \nu\Delta w + \Gamma(H, Q, \ell, \omega) + (w \cdot \nabla)v^S + (v^S \cdot \nabla)w + \nabla q = F(\mathbf{X}, q) \quad \text{in } (0, +\infty) \times \mathcal{F}, \tag{5.1}$$

$$\operatorname{div} w = 0 \quad \text{in } (0, +\infty) \times \mathcal{F}, \tag{5.2}$$

$$w = \ell + \omega \times (y - h^S) \quad \text{on } (0, +\infty) \times \partial\mathcal{S}, \tag{5.3}$$

$$w = u \quad \text{on } (0, +\infty) \times \partial\Omega, \tag{5.4}$$

$$M\ell' = -\int_{\partial\mathcal{S}} \mathbb{T}(w, q)n \, d\Gamma + Q^* f_M^S + \varepsilon_\ell(\mathbf{X}), \quad t > 0, \tag{5.5}$$

$$I_S\omega' = -\int_{\partial\mathcal{S}} (y - h^S) \times \mathbb{T}(w, q)n \, d\Gamma + Q^* f_I^S + \varepsilon_\omega(\mathbf{X}), \quad t > 0, \tag{5.6}$$

$$H' = \ell + \varepsilon_H(\mathbf{X}), \quad t > 0, \tag{5.7}$$

$$Q' = \mathbb{S}(\omega) + \varepsilon_Q(\mathbf{X}), \quad t > 0, \tag{5.8}$$

$$H(0) = h^0 - h^S, \quad Q(0) = R^0(R^S)^* - \mathrm{Id}, \quad \ell(0) = \ell^0, \quad \omega(0) = \omega^0, \quad w(0, y) = w^0(y) \quad y \in \mathcal{F}. \tag{5.9}$$

In the above system, we write

$$\mathbf{X} \stackrel{\text{def}}{=} \begin{bmatrix} w \\ \ell \\ \omega \\ H \\ Q \end{bmatrix} \tag{5.10}$$

$$I_S \stackrel{\text{def}}{=} I(h^S, R^S), \quad \text{(see (1.7))}$$

and

$$\varepsilon_H(\mathbf{X}) \stackrel{\text{def}}{=} Q\ell, \quad \varepsilon_Q(\mathbf{X}) \stackrel{\text{def}}{=} Q\mathbb{S}(\omega), \quad \varepsilon_\ell(\mathbf{X}) \stackrel{\text{def}}{=} -M\omega \times \ell, \quad \varepsilon_\omega(\mathbf{X}) \stackrel{\text{def}}{=} I_S\omega \times \omega,$$

$$\begin{aligned} F(\mathbf{X}, q) \stackrel{\text{def}}{=} &- [\mathbf{N}^S(w)] + [(\mathrm{Id} - \mathbf{K})\partial_t w] + \nu[(\mathbf{L} - \Delta)w] - [\mathbf{M}w] + [(\nabla - \mathbf{G})q] \\ &- (\varepsilon_L(H, Q) + \varepsilon_M(H, Q, \ell, \omega) + \varepsilon_N(H, Q) + \varepsilon_G(H, Q) - \varepsilon_f(H, Q)), \end{aligned}$$

where Γ, ε_L, ε_M, ε_N, ε_G and ε_f are defined similarly as in Proposition 12 and Proposition 13.

Note that to obtain the above system we have used the following relations

$$I(h(t), R(t)) = R(t)(R^S)^* I_S R^S (R(t))^* \quad \text{and} \quad Ua \times Ub = U(a \times b) \quad \forall a, b \in \mathbb{R}^3, \ U \in SO(3).$$

A first step in the proof of the stabilization of the above system consists in - as in the 2D case - to consider a linear system associated to (5.1)-(5.9). We set

$$\begin{aligned} \mathcal{H} \stackrel{\text{def}}{=} \Big\{ &[w, \ell, \omega, H, Q]^* \in \mathbf{L}^2(\mathcal{F}) \times \mathbb{R}^9 \times \mathbb{R}^{3\times 3} \ ; \ w \cdot n = (\ell + \omega \times (y - h^S)) \cdot n \text{ on } \partial\mathcal{S}, \\ &w \cdot n = 0 \text{ on } \partial\Omega, \ \operatorname{div} w = 0 \quad \text{in } \mathcal{F} \Big\}, \end{aligned}$$

$$\mathcal{V} \stackrel{\text{def}}{=} \Big\{ [w, \ell, \omega, H, Q]^* \in \mathcal{H} \ ; \ w \in \mathbf{H}^1(\mathcal{F}), \ w = \ell + \omega \times (y - h^S) \text{ on } \partial\mathcal{S}, \ w = 0 \text{ on } \partial\Omega \Big\}.$$

We define the linear operator $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ as follows: we set

$$\mathcal{D}(A) \stackrel{\text{def}}{=} \{[w, \ell, \omega, H, Q]^* \in \mathcal{V} \ ; \ w \in \mathbf{H}^2(\mathcal{F})\}, \qquad A \stackrel{\text{def}}{=} P\widetilde{A},$$

where for $\left[w, \ell, \omega, H, Q\right]^* \in \mathcal{D}(A)$, we set

$$\widetilde{A}\begin{bmatrix} w \\ \ell \\ \omega \\ H \\ Q \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \nu\Delta w - \Gamma(H, Q, \ell, \omega) - (w \cdot \nabla)v^S - (v^S \cdot \nabla)w \\ -M^{-1}\displaystyle\int_{\partial\mathcal{S}} 2\nu D(w)n \, d\Gamma + M^{-1}Q^* f_M^S \\ -I_S^{-1}\displaystyle\int_{\partial\mathcal{S}} (y - h^S) \times 2\nu D(w)n \, d\Gamma + I_S^{-1}Q^* f_I^S \\ \ell \\ \mathbb{S}(\omega) \end{bmatrix}$$

and where P is the orthogonal projection from $\mathbf{L}^2(\mathcal{F}) \times \mathbb{R}^9 \times \mathbb{R}^{3\times 3}$ onto $\mathcal{H}$. We define the control operator $B \in \mathcal{L}(\mathbf{V}^0(\partial\Omega), [\mathcal{D}(A^*)]')$ in a similar way as in the 2D case (see (3.26)). Then the nonhomogeneous linear system corresponding to (5.1)-(5.9) rewrite as (3.11)-(3.12) where $\mathbf{X}$ is given by (5.10).

In a way completely similar to the proof of Proposition 21, for $\sigma > 0$ we can prove that $(A + \sigma, B)$ is stabilizable by a finite dimensional control, namely, we can find families $[\widetilde{\varphi}_j, \widetilde{\xi}_j, \widetilde{\zeta}_j, \widetilde{a}_j, \widetilde{c}_j]^* \in \mathcal{D}(A^*)$ and $v_j \in \mathbf{V}^{\frac{3}{2}}(\partial\Omega)$, $j = 1, \dots, N_\sigma$, and a corresponding feedback operator $F_\sigma : \mathcal{H} \to \mathbf{V}^{\frac{3}{2}}(\partial\Omega)$ defined by

$$F_\sigma[w, \ell, \omega, H, Q]^* = \sum_{j=1}^{N_\sigma} \left(\int_{\mathcal{F}} w \cdot \widetilde{\varphi}_j dy + M\ell \cdot \widetilde{\xi}_j + I_S\omega \cdot \widetilde{\zeta}_j + H \cdot \widetilde{a}_j + Q : \widetilde{c}_j \right) v_j$$

such that the linear operator $A_\sigma \stackrel{\text{def}}{=} A + BF_\sigma$ with domain $\mathcal{D}(A_\sigma) \stackrel{\text{def}}{=} \{\mathbf{X} \in \mathcal{H} \mid A\mathbf{X} + BF_\sigma\mathbf{X} \in \mathcal{H}\}$ is the infinitesimal generator of an analytic and exponentially stable semigroup on $\mathcal{H}$ of type lower than $-\sigma$. However, such a feedback law does not permit to construct a fixed-point solution for the nonlinear system similarly as in Section 3.3, because, as explained in the introduction, we do not necessarily have $F_\sigma[w, \ell, \omega, H, Q]^*$ equal to the trace of w_0 on $\partial\Omega$, which is required to defined a solution of the 3D problem.

Then the main change with respect to the 2D case is that, in order to guarantee the initial compatibility condition $u(0) = w_0$ on $\partial\Omega$ that will allow to construct a strong solution, we suppose that $w_0 \in \mathbf{H}^1(\Omega)$ satisfies $w_0 = 0$ on $\partial\Omega$ and we complete the system (5.1)-(5.9) with a dynamical equation for u:

$$u(t, x) = \Phi\overline{u} \stackrel{\text{def}}{=} \sum_{j=1}^{N_\sigma} u_j v_j, \tag{5.11}$$

with $\overline{u} \stackrel{\text{def}}{=} (u_j)_{j\in\{1,\dots,N_\sigma\}}$ satisfying

$$\overline{u}' = \overline{g}, \quad \overline{u}(0) = 0. \tag{5.12}$$

Then with (5.10) and by setting $\mathbf{F}(\mathbf{X}, q) \stackrel{\text{def}}{=} [F(\mathbf{X}, q), \varepsilon_\ell(\mathbf{X}), \varepsilon_\omega(\mathbf{X}), \varepsilon_H(\mathbf{X}), \varepsilon_Q(\mathbf{X})]^*$ the nonlinear system (5.1)-(5.9), (5.11) and (5.12) rewrites

$$\begin{array}{rcl} P\mathbf{X}' &=& AP\mathbf{X} + B\Phi\overline{u} + P\mathbf{F}(\mathbf{X}, q) \ \text{ in } [\mathcal{D}(A^*)]', \quad P\mathbf{X}(0) = P\mathbf{X}^0, \\ \overline{u}' &=& \overline{g}, \quad \overline{u}(0) = 0, \\ (I - P)\mathbf{X} &=& (I - P)D_{\mathcal{F}}\Phi\overline{u}. \end{array} \tag{5.13}$$

Note that to define a strong solution to (5.1)–(5.9), (5.11) and (5.12) (i.e. a solution such that $t \mapsto w(t)$ is continuous with values in $\mathbf{H}^1(\Omega)$) we have to assume $P\mathbf{X}^0 \in \mathcal{V}$. It can be easily seen that this last condition is satisfied if v^0 (i.e. the initial velocity of the system before the change of variables) belongs to $\mathbf{H}^1(\mathcal{F}(h^0, R^0))$ and satisfies the compatibility conditions (1.20), (1.21) and (1.22).

Thus, by following the general framework of [5] for the construction of dynamical control (see also [7]) we introduce the "extended" space $\mathcal{H}_E \stackrel{\text{def}}{=} \mathcal{H} \times \mathbb{R}^{N_\sigma}$, the following extended operator on $\mathcal{H}_E$:

$$\begin{array}{rcl} A_E &\stackrel{\text{def}}{=}& \begin{bmatrix} A & B\Phi \\ 0 & 0 \end{bmatrix}, \\ D(A_E) &\stackrel{\text{def}}{=}& \{(\mathbf{Y}, \overline{u}) \in \mathcal{H}_E \mid A\mathbf{Y} + B\Phi\overline{u} \in \mathcal{H}\}, \end{array}$$

as well as the control operator $B_E \in \mathcal{L}(\mathbb{R}^{N_\sigma}, \mathcal{H}_E)$:

$$B_E \overline{g} \stackrel{\text{def}}{=} \begin{bmatrix} 0 \\ \overline{g} \end{bmatrix}.$$

By setting $\mathbf{Y}_E^0 = [P\mathbf{X}^0, 0]^*$, $\mathbf{Y}_E = [P\mathbf{X}, \overline{u}]^*$ and $\mathbf{F}_E(\mathbf{Y}_E, q) \stackrel{\text{def}}{=} [P\mathbf{F}(P\mathbf{X} + (I-P)D_{\mathcal{F}}\Phi\overline{u}, q), 0]^*$, system (5.13) reduces to

$$\mathbf{Y}_E' = A_E \mathbf{Y}_E + B_E \overline{g} + \mathbf{F}_E(\mathbf{Y}_E, q), \quad \mathbf{Y}_E(0) = \mathbf{Y}_E^0.$$

Thus, since the stabilizability of $(A+\sigma, B)$ by finite dimensional control generated by the family (v_j) implies the stabilizability of $(A_E + \sigma, B_E)$ (see [5, Theorem 8]), we can find a family $[\varphi_j, \xi_j, \zeta_j, a_j, c_j]^* \in \mathcal{D}(A^*)$, a matrix Λ of size $N_\sigma \times N_\sigma$ and a corresponding feedback operator $F_{E,\sigma} : \mathcal{H}_E \to \mathbb{R}^{N_\sigma}$ defined by

$$F_{E,\sigma}[w, \ell, \omega, H, Q, \overline{u}]^* = \Lambda \overline{u} + \sum_{j=1}^{N_\sigma} \left(\int_{\mathcal{F}} w \cdot \varphi_j dy + M\ell \cdot \xi_j + I_S \omega \cdot \zeta_j + H \cdot a_j + Q : c_j \right) e_j$$

where $\{e_j \mid j = 1, \ldots, N_\sigma\}$ is a given basis of $\mathbb{R}^{N_\sigma}$, such that the linear operator $A_{E,\sigma} \stackrel{\text{def}}{=} A_E + B_E F_{E,\sigma}$ with domain $\mathcal{D}(A_{E,\sigma}) = \mathcal{D}(A_E)$ (since B_E is bounded in $\mathcal{H}_E$) is the infinitesimal generator of an analytic and exponentially stable semigroup on $\mathcal{H}_E$ of type lower than $-\sigma$.

Finally, using a fixed point argument, we can prove as for the 2D case that for small initial conditions $P\mathbf{X}^0$ in $\mathcal{V}$, this feedback operator stabilizes the system (5.1)–(5.9), (5.11) and (5.12) with $\overline{g} = F_{E,\sigma}[w, \ell, \omega, H, Q, \overline{u}]^*$. Writing this feedback operator in the initial variables gives (1.19).

References

[1] C. Amrouche and V. Girault. *Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension,* Czechoslovak Math. J., 44 (1994), 109–140.

[2] M. Badra. *Feedback stabilization of the 2-D and 3-D Navier-Stokes equations based on an extended system,* ESAIM Control Optim. Calc. Var., 15 (2009), 934–968.

[3] M. Badra. *Lyapunov function and local feedback boundary stabilization of the Navier-Stokes equations*, SIAM J. Control Optim., 48 (2009), 1797–1830.

[4] M. Badra. *Abstract settings for stabilization of nonlinear parabolic system with a Riccati-based strategy. Application to Navier-Stokes and Boussinesq equations with Neumann or Dirichlet control,* Discrete Contin. Dyn. Syst., 32 (2012), 1169–1208.

[5] M. Badra and T. Takahashi. *Stabilization of parabolic nonlinear systems with finite dimensional feedback or dynamic controllers. Application to Navier-Stokes system,* SIAM J. Control Optim., 49 (2011), 420–463.

[6] M. Badra and T. Takahashi. *Feedback stabilization of a simplified 1d fluid–particle system,* Ann. Inst. H. Poincaré Anal. Non Linéaire, 31 (2014), 369–389.

[7] M. Badra and T. Takahashi. *On Fattorini criterion for approximate controllability and stabilizability of parabolic equations,* ESAIM Control Optim. Calc. Var., to appear.

[8] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. K. Mitter. *Representation and control of infinite dimensional systems,* Systems & Control: Foundations & ApplicationsḂirkhäuser Boston Inc., Boston, MA, second edition (2007).

[9] M. Boulakia and S. Guerrero. *Local null controllability of a fluid−solid interaction problem in dimension 3,* J. Eur. Math. Soc., 15 (2013), 825–856.

[10] M. Boulakia and A. Osses. *Local null controllability of a two-dimensional fluid-structure interaction problem,* ESAIM Control Optim. Calc. Var., 14 (2008), 1–42.

[11] M. Boulakia, E. L. Schwindt, and T. Takahashi. *Existence of strong solutions for the motion of an elastic structure in an incompressible viscous fluid,* Interfaces Free Bound., 14 (2012), 273–306.

[12] C. Conca, J. A. San Martín, and M. Tucsnak. *Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid,* Comm. Partial Differential Equations, 25 (2000), 1019–1042.

[13] R. Denk, M. Hieber, and J. Prüss. *$\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type,* Mem. Amer. Math. Soc., 166 (2003), viii+114.

[14] B. Desjardins and M. J. Esteban. *Existence of weak solutions for the motion of rigid bodies in a viscous fluid,* Arch. Ration. Mech. Anal., 146 (1999), 59–71.

[15] A. Doubova and E. Fernández-Cara. *Some control results for simplified one-dimensional models of fluid-solid interaction,* Math. Models Methods Appl. Sci., 15 (2005), 783–824.

[16] C. Fabre and G. Lebeau. *Prolongement unique des solutions de l'equation de Stokes,* Comm. Partial Differential Equations, 21 (1996), 573–596.

[17] E. Feireisl. *On the motion of rigid bodies in a viscous fluid,* Appl. Math., 47 (2002), 463–484.

[18] H. Fujita and H. Morimoto. *On fractional powers of the Stokes operator,* Proc. Japan Acad., 46 (1970), 1141–1143.

[19] O. Glass and F. Sueur. *Uniqueness results for weak solutions of two-dimensional fluid-solid systems,* preprint.

[20] C. Grandmont and Y. Maday. *Existence for an unsteady fluid-structure interaction problem,* M2AN Math. Model. Numer. Anal., 34 (2000), 609–636.

[21] P. Grisvard. *Caractérisation de quelques espaces d'interpolation,* Arch. Rational Mech. Anal., 25 (1967), 40–63.

[22] M. D. Gunzburger, H.-C. Lee, and G. A. Seregin. *Global existence of weak solutions for viscous incompressible flows around a moving rigid body in three dimensions,* J. Math. Fluid Mech., 2 (2000), 219–266.

[23] O. Yu. Imanuvilov and T. Takahashi. *Exact controllability of a fluid-rigid body system,* J. Math. Pures Appl., 87 (2007), 408–437.

[24] A. Inoue and M. Wakimoto. *On existence of solutions of the Navier-Stokes equation in a time dependent domain,* J. Fac. Sci. Univ. Tokyo Sect. IA Math., 24 (1977), 303–319.

[25] J. Lequeurre. *Null Controllability of a Fluid-Structure System,* SIAM J. Control Optim., 51 (2013), 1841–1872.

[26] Y. Liu, T. Takahashi, and M. Tucsnak. *Single input controllability of a simplified fluid-structure interaction model,* ESAIM Control Optim. Calc. Var., 19 (2013), 20–42.

[27] J.-P. Raymond. *Feedback boundary stabilization of the three-dimensional incompressible Navier-Stokes equations,* J. Math. Pures Appl., 87 (2007), 627–669.

[28] J.-P. Raymond. *Feedback stabilization of a fluid-structure model,* SIAM J. Control Optim., 48 (2010), 5398–5443.

[29] J.-P. Raymond and T. Thevenet. *Boundary feedback stabilization of the two dimensional Navier-Stokes equations with finite dimensional controllers,* Discrete and Continuous Dynamical Systems A, 27 (2010), 1159–1187.

[30] J.-P. Raymond and M. Vanninathan. *Null controllability in a heat-solid structure model,* Appl. Math. Optim., 59 (2009),247–273.

[31] J.-P. Raymond and M. Vanninathan. *Null controllability in a fluid-solid structure model,* J. Differential Equations, 248 (2010), 1826–1865.

[32] Jean-Pierre Raymond. *Feedback boundary stabilization of the two-dimensional Navier-Stokes equations,* SIAM J. Control Optim., 45 (2006), 790–828.

[33] J. A. San Martín, V. Starovoitov, and M. Tucsnak. *Global weak solutions for the two-dimensional motion of several rigid bodies in an incompressible viscous fluid,* Arch. Ration. Mech. Anal., 161 (2002), 113–147.

[34] T. Takahashi. *Analysis of strong solutions for the equations modeling the motion of a rigid-fluid system in a bounded domain,* Adv. Differential Equations, 8 (2003), 1499–1532.

[35] T. Takahashi and M. Tucsnak. *Global strong solutions for the two-dimensional motion of an infinite cylinder in a viscous fluid,* J. Math. Fluid Mech., 6 (2004), 53–77.

[36] R. Temam. *Navier-Stokes equations. Theory and numerical analysis.* North-Holland Publishing Co., Amsterdam, Studies in Mathematics and its Applications, Vol. 2 (1977).