DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement 2011

Website: www.aimSciences.org

pp. **117–125**

EXISTENCE RESULTS TO A QUASILINEAR AND SINGULAR PARABOLIC EQUATION

Mehdi Badra, Kaushik Bal and Jacques Giacomoni

LMAP (UMR 5142), Bat. IPRA
Université de Pau et des Pays de l'Adour
Avenue de l'Université, 64013 cedex Pau, France

Abstract. We investigate the following quasilinear parabolic and singular equation,

$$\begin{cases} u_t - \Delta_p u = \dfrac{1}{u^\delta} + f(t,x) \text{ in } Q_T = (0,T] \times \Omega \\ u > 0 \text{ in } Q_T \text{ , } u = 0 \text{ on } \Gamma = [0,T] \times \partial\Omega, \\ u(0,x) = u_0(x) \text{ in } \Omega \end{cases} \tag{P$_t$}$$

where Ω is an open bounded domain with smooth boundary in R^N, $1 < p < \infty$ and $0 < \delta$, $T > 0$, $f \in L^\infty(Q_T)$ and $u_0 \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$. For any $\delta \in (0, 2 + \frac{1}{p-1})$, u_0 satisfying a cone condition defined below and any $T > 0$, In this paper we prove the existence and the uniqueness of a weak solution u to ($\mathrm{P_t}$).

1. **Introduction.** In the present paper we investigate the following quasilinear and singular parabolic problem :

$$\begin{cases} u_t - \Delta_p u = \dfrac{1}{u^\delta} + f(t,x) \text{ in } Q_T = (0,T] \times \Omega \\ u > 0 \text{ in } Q_T \text{ , } u = 0 \text{ on } \Gamma = [0,T] \times \partial\Omega, \\ u(0,x) = u_0(x) \text{ in } \Omega \end{cases} \tag{P$_t$}$$

where Ω is an open bounded domain with smooth boundary in R^N (with $N \geq 2$), $1 < p < \infty$ and $0 < \delta$, $T > 0$, $f \in L^\infty(Q_T)$ is nonnegative and $u_0 \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$. Such problems arise in different models: non newtonien flows, chemical hetereogeneous catalyst kinetics, combustion. We refer to the survey Hernández-Mancebo-Vega [14], the book Ghergu-Radulescu [11] and therein the bibliography for more details about the corresponding models.

Our goal in this paper is to prove the existence and the uniqueness of the weak solution to (P_t) defined as follows:

Definition 1.1.

$$\mathbf{V}(Q_T) = \{u \,:\, u \in L^\infty(Q_T),\, u_t \in L^2(Q_T),\, u \in L^\infty(0,T; W_0^{1,p}(\Omega))\}$$

and

2000 *Mathematics Subject Classification.* Primary 35J65, 35J20; Secondary 35J70 .

Key words and phrases. quasilinear parabolic equation, singular nonlinearity, sub and supersolutions.

Definition 1.2. A *weak* solutions to (P_t) is a function $u \in \mathbf{V}(Q_T)$ satisfying

1. for any compact $K \in Q_T$, $essinf_K u > 0$,
2. for every test function $\phi \in \mathbf{V}(Q_T)$,

$$\int_{Q_T} \left[\phi \frac{\partial u}{\partial t} - |\nabla u|^{p-2}\nabla u \nabla \phi - \phi(\frac{1}{u^\delta} + f(t,x))\right] dz = 0, \; z = (t,x),$$

3. for every $\phi(x) \in W_0^{1,p}(\Omega)$

$$\int_\Omega \phi(x) u(t,x) \to \int_\Omega \phi(x) u_0(x) \text{ as } t \to 0^+.$$

The approach we use is to study first the existence of solutions to the stationnary problem (P) that is for $g \in L^\infty(\Omega)$, $\lambda > 0$

$$(\mathrm{P}) \begin{cases} u - \lambda(\Delta_p u + \frac{1}{u^\delta}) = g \text{ in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases}$$

To control the singular term $\frac{1}{u^\delta}$, we need to consider solutions in the cone $\mathcal{C}$ where $\mathcal{C}$ is the set of functions $v \in L^\infty(\Omega)$ such that $\exists\, c_1, c_2 > 0$ with

$$\begin{cases} c_1 d(x) \le v \le c_2 d(x) \;\text{ if }\; \delta < 1, \\ c_1 d(x) log^{\frac{1}{p}}(\frac{k}{d(x)}) \le v \le c_2 d(x) log^{\frac{1}{p}}(\frac{k}{d(x)}) \;\text{ with } k \text{ large and if }\; \delta = 1, \\ c_1 d(x)^{\frac{p}{\delta+p-1}} \le v \le c_2 d(x)^{\frac{p}{\delta+p-1}} \;\text{ if }\; \delta > 1. \end{cases}$$

with $d(x) = \operatorname{dist}(x, \partial\Omega)$. Regarding Problem (P), we prove the following results :

Theorem 1.3. *Let $g \in L^\infty(\Omega)$ and $0 < \delta < 2 + \frac{1}{p-1}$. Then for any $\lambda > 0$, there exists a unique u_λ in $W_0^{1,p}(\Omega) \cap \mathcal{C}$ such that*

$$\begin{cases} u - \lambda(\Delta_p u + \frac{1}{u^\delta}) = g \text{ in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases}$$

Concerning the case where $\delta \ge 2 + \frac{1}{p-1}$, we prove that

Theorem 1.4. *Let $g \in L^\infty(\Omega)$ and $0 < \delta \ge 2 + \frac{1}{p-1}$. Then for any $\lambda > 0$, there exists u in $W_{loc}^{1,p}(\Omega) \cap \mathcal{C}$ such that*

$$\begin{cases} u - \lambda(\Delta_p u + \frac{1}{u^\delta}) = g \text{ in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases}$$

Furthermore, $u \notin W_0^{1,p}(\Omega)$.

Using a time discretization method, Theorem 1.3, energy estimates and the weak comparison principle (see CUESTA-TAKÁČ [6], FLECKINGER-TAKÁČ [10]), we prove the following result concerning (P_t) :

Theorem 1.5. *Let $0 < \delta < 2 + \frac{1}{p-1}$, $f \in L^\infty(Q_T)$ and $u_0 \in W_0^{1,p}(\Omega) \cap \mathcal{C}$. Then there exists a unique weak solution u to (P_t) such that $u(t) \in \mathcal{C}$ uniformly for $t \in [0,T]$.*

Remarks.

1. Since $u \in \mathbf{V}(Q_T)$, it follows that $u \in C(0,T; L^2(\Omega))$.
2. By Theorem 1.4, the restriction $\delta < 2 + \frac{1}{p-1}$ is sharp.

We give now briefly the state of art concerning parabolic quasilinear singular equations. The corresponding stationnary equation was studied profusely in the litterature. In particular the case $p = 2$, mostly when $\delta < 1$ and also when g depends on u was considered in detail (see the pionniering work CRANDALL-RABINOWITZ-TARTAR [5], the bibliography in HERNÁNDEZ-MANCEBO [13] and PERERA-SILVA [15]). The case $p \neq 2$ was not considerd so far. We can mention the work ARANDA-GODOY [3] where existence results are obtained via the bifurcation theory for $1 < p \leq 2$ and $g = g(u)$ satisfying some growth conditions. In GIACOMONI-SCHINDLER-TAKÁČ [12] the existence and multiplicity results (for $1 < p < \infty$ $g(u) = u^q$ with $1 < q \leq p^* - 1$ and $0 < \delta < 1$ are proved by using variational methods and regularity results in Hölder spaces. Concerning the parabolic case, avalaible results mostly concern the case $p = 2$. In this regard, we can quote the result in HERNANDEZ-MANCEBO-VEGA [14] where in the range $0 < \delta < \frac{1}{2}$, properties of the linearised operator (in $C_0^1(\bar{\Omega})$) and the validity of the strong maximum principle are studied. We also mention the work DAVILA-MONTENEGRO [7] still concerning the case $p = 2$ and with singular absorption term. In this nice work, the authors achieved uniqueness within the class of functions satisfying $u(x,t) \geq c(\mathrm{dist}(x, \partial\Omega)^\gamma$ for suitable γ and $c > 0$ and discuss the asymptotic behaviour of solutions. Finally, we would like to quote the beautiful paper WINKLER [19] where the author show that uniqueness is violated in case of non homogeneous boundary Dirichlet condition.

2. **Proof of Theorems 1.3 and 1.4.** We begin by proving Theorem 1.3.

Proof. First, let us consider the case $\delta < 1$. For $\lambda > 0$, we then define the following energy functional :

$$E_\lambda(u) = \frac{1}{2}\int_\Omega u^2 + \frac{\lambda}{p}\int_\Omega |\nabla u|^p - \frac{\lambda}{1-\delta}\int_\Omega (u^+)^{1-\delta} - \int_\Omega gu$$

E_λ is well defined in $X = W_0^{1,p}(\Omega)$ if $p \geq \frac{2n}{n+2}$. If $1 < p < \frac{2n}{n+2}$, E_λ is well defined in $X = W_0^{1,p}(\Omega) \cap L^2(\Omega)$. It is easy to see that E_λ is stricly convex, continuous and coercive in X. Thus, since X is reflexive, E_λ admits a unique global minimizer, we denote by u_λ. We show now that $u_\lambda \in \mathcal{C}$. Let ϕ_1 be the positive normalized eigenfunction associated to the first eigenvalue $\lambda_1(\Omega)$ of $-\Delta_p$ with homogeneous boundary Dirichlet conditions (see ANANE [1], [2] for further details). Next, we observe that for $\epsilon > 0$ small enough (depending on λ, δ and g) we have

$$\begin{cases} \epsilon\phi_1 - \lambda(\Delta_p(\epsilon\phi_1) + \frac{1}{(\epsilon\phi_1)^\delta}) < g & \text{in } \Omega \\ \epsilon\phi_1|_{\partial\Omega} = 0. \end{cases}$$

Then, for $t > 0$, let $v_\lambda = (\epsilon\phi_1 - u_\lambda)^+$ and $\chi(t) = E_\lambda(u_\lambda + tv_\lambda)$. From Hardy Inequality, it follows that χ is Gâteaux-differentiable for $t \in (0, 1]$ and

$$\chi'(t) =< E'_\lambda(u_\lambda + tv_\lambda), v_\lambda > .$$

Since E_λ is strictly convex, we get that $t \to \chi'(t)$ is increasing. Therefore, for $0 < t < 1$ small enough, we obtain that

$$0 \leq \chi'(t) < \chi'(1) =< E'_\lambda(\epsilon\phi_1), v_\lambda >< 0$$

if v_λ has non-zero measure support. Thus, $\epsilon\phi_1 \leq u_\lambda$ and E_λ is Gâteaux-differentiable in u_λ. Consequently, for any $\phi \in X$,

$$< E'_\lambda(u_\lambda), \phi >=< u_\lambda - \lambda(\Delta_p u_\lambda + \frac{1}{u_\lambda^\delta}) - g, \phi > .$$

By the weak comparison principle, we have also that

$$u_\lambda \leq M$$

for any $M > |g|_{L^\infty(\Omega)} + \frac{\lambda}{|g|^\delta_{L^\infty(\Omega)}}$.Then, $u_\lambda \in L^\infty(\Omega)$. Let $U_\lambda \in C^{1,\alpha}(\bar{\Omega})$ (with suitable $0 < \alpha < 1$) be the unique positive solution (see GIACOMONI-SCHINDLER-TAKÁČ [12]) to

$$\begin{cases} -\Delta_p u = \frac{\lambda}{u^\delta} \quad \text{in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases}$$

Then, again by the weak comparison principle there exists $M' > 0$ large enough such that $u_\lambda \leq M' U_\lambda$ from which it follows that $u_\lambda \in \mathcal{C}$. We consider now the case $\delta \geq 1$. We use in this case the method of sub and supersolution and the following approximated problem :

$$(\mathrm{P}_\epsilon) \begin{cases} u - \lambda(\Delta_p u + \frac{1}{(u+\epsilon)^\delta}) = f \quad \text{in } \Omega \\ u|_{\partial\Omega} = 0, \ u > 0 \quad \text{in } \Omega. \end{cases}$$

For $\delta = 1$, by straightforward computations we have that for $A > 0$ (depending of the diameter of Ω and large enough, $\eta > 0$ small enough (depending of λ and g but not of ϵ)

$$\underline{u}_\epsilon = (\eta\phi_1 + \epsilon')[\ln(\frac{A}{\eta\phi_1 + \epsilon'}]^{\frac{1}{p}} - \epsilon'[\ln(\frac{A}{\epsilon'})]^{\frac{1}{p}},$$

with ϵ' satisfying $\epsilon = \epsilon'[\ln(\frac{A}{\epsilon'})]^{\frac{1}{p}}$, is a subsolution to (P_ϵ). Similarly, for $M > 0$ large enough (depending of λ and g but not of ϵ)

$$\bar{u}_\epsilon = (M\phi_1 + \epsilon')[\ln(\frac{A}{M\phi_1 + \epsilon'})]^{\frac{1}{p}} - \epsilon'[\ln(\frac{A}{\epsilon'})]^{\frac{1}{p}},$$

is a supersolution to (P_ϵ) satisfying $\bar{u}_\epsilon \geq \underline{u}_\epsilon$. If $\delta > 1$, we consider the following subsolution and supersolution respectively:

$$\underline{u}_\epsilon = \eta[(\phi_1 + \epsilon^{\frac{p-1+\delta}{p}})^{\frac{p}{p-1+\delta}} - \epsilon],$$

for $\eta > 0$ small enough and

$$\bar{u}_\epsilon = M[(\phi_1 + \epsilon^{\frac{p-1+\delta}{p}})^{\frac{p}{p-1+\delta}} - \epsilon],$$

for $M > 0$ large enough. Since the operator $u \to -\Delta_p u - \frac{1}{(u+\epsilon)^\delta}$ is monotone from $(W^{1,p}_0(\Omega))^+$ to $W^{-1,\frac{p}{p-1}}(\Omega)$ (see DEIMLING [8]), we get from the sub and supersolution technique that there exists a unique solution u_ϵ to (P_ϵ) such that

$$\underline{u}_\epsilon \leq u_\epsilon \leq \bar{u}_\epsilon. \tag{1}$$

From the strong comparison principle of [6], we have that

$$0 < \epsilon_1 < \epsilon_2 \Rightarrow \begin{cases} u_{\epsilon_2} < u_{\epsilon_1} \quad \text{in } \Omega \\ u_{\epsilon_1} + \epsilon_1 < u_{\epsilon_2} + \epsilon_2 \quad \text{in } \Omega \end{cases}$$

from which it follows that $(u_{\epsilon_n})_{n\in\mathrm{N}}$ is a Cauchy sequence as $\epsilon_n \to 0^+$ in $C_0(\bar{\Omega})$. Now, if $\delta < 2 + \frac{1}{p-1}$, we get from (1) and the Hardy Inequality that

$$\limsup_{n\in\mathrm{N}} \int_\Omega \frac{u_{\epsilon_n}}{(u_{\epsilon_n} + \epsilon_n)^\delta} < +\infty.$$

Passing to the limit as $\epsilon_n \to 0^+$, we get the existence of $u \in W_0^{1,p}(\Omega)$ such that

$$(\eta\phi_1)[\ln(\frac{A}{\eta\phi_1}]^{\frac{1}{p}} \le u \le (M\phi_1)[\ln(\frac{A}{M\phi_1}]^{\frac{1}{p}} \quad \text{if } \delta = 1,$$

$$\eta(\phi_1)^{\frac{p}{p-1+\delta}} \le u \le M(\phi_1)^{\frac{p}{p-1+\delta}} \quad \text{if } \delta > 1$$

from which it follows that $u \in \mathcal{C}$. Then it is easy to derive that u is a weak solution to (P). The uniqueness of the solution to (P) in $W_0^{1,p}(\Omega) \cap \mathcal{C}$ follows from the monotonicity of $u \to -\Delta_p u - \frac{1}{u^\delta}$ in $W_0^{1,p}(\Omega) \cap \mathcal{C}$. □

We prove now Theorem 1.4 :

Proof. Let $\delta \ge 2 + \frac{1}{p-1}$. We give an alternative proof for existence of solutions. Let $(\Omega_k)_k$ be an increasing sequence of smooth domains such that $\Omega_k \uparrow \Omega$ and $\text{dist}(x, \partial\Omega) \ge \frac{1}{k}\ \forall\, x \in \Omega_k$. We use the sub and super solutions technique in Ω_k and pass to the limit as $k \to \infty$. For $\eta > 0$ and $M > \eta$, let

$$\underline{u} = \eta(\phi_1)^{\frac{p}{p-1+\delta}}, \quad \bar{u} = M(\phi_1)^{\frac{p}{p-1+\delta}}.$$

For η small enough and M large enough, $\underline{u}$ and $\bar{u}$ are respectively a subsolution and a supersolution to (P). Then, there exists u_k solution to

$$\begin{cases} u - \lambda(\Delta_p u + \frac{1}{u^\delta}) = g \quad \text{in } \Omega_k \\ u|_{\partial\Omega_k} = \underline{u} \end{cases}$$

and satisfying $\underline{u} \le u_k \le \bar{u}$. From the weak comparison principle, we have that $u_k \le u_{k+1}$ in Ω_k. Using local regularity results (see Serrin [16], Tolksdorf [18] and [17], DiBenetdetto [4]), we get that $\underline{u} \le u = \lim_{k\to\infty} u_k \in W^{1,p}_{\text{loc}}(\Omega)$ and satisfies (P) in the sense of distributions. Let us show that $u \notin W_0^{1,p}(\Omega)$. For that, we argue by contradiction: assume that $u \in W_0^{1,p}(\Omega)$. Then, from the equation in (P), we get that $\frac{1}{u^\delta} \in W^{-1,\frac{p}{p-1}}(\Omega)$. Thus, $\int_\Omega \bar{u}^{1-\delta} \le \int_\Omega u^{1-\delta} < +\infty$ which is a contradiction by definition of $\bar{u}$. the proof of Theorem 1.4 is now complete. □

3. **Proof of Theorem 1.5.** Using Theorem 1.3 and a time discretization method, we prove Theorem 1.5:

Proof. Let $N \in$ N, $n \ge 2$ and $\Delta_t = \frac{T}{N}$. For $0 \le n \le N$, we define $t_n = n\Delta_t$, $f^n(\cdot) = \frac{1}{\Delta_t}\int_{(n-1)\Delta_t}^{n\Delta_t} f(s,\cdot)\,\mathrm{d}s \in L^\infty(\Omega)$. From Theorem 1.3 (with $\lambda = \Delta_t$, $g = \Delta_t f^n + u^{n-1} \in L^\infty(\Omega)$), we define by iteration $u^n \in W_0^{1,p}(\Omega) \cap \mathcal{C}$ with the following scheme:

$$\begin{cases} \frac{u^n - u^{n-1}}{\Delta_t} - \Delta_p u^n - \frac{1}{(u^n)^\delta} = f^n \quad \text{in } \Omega \\ u^n|_{\partial\Omega} = 0 \end{cases} \tag{2}$$

and $u^0 = u_0 \in W_0^{1,p}(\Omega) \cap \mathcal{C}$. Then, defining the following functions :

$$\left[\begin{array}{c} u_{\Delta_t}(t) = u^n \quad \text{for } t \in [(n-1)\Delta_t, n\Delta_t),\ \forall\, n \in [1, \cdots, N], \\ \tilde{u}_{\Delta_t}(t) = \frac{(t-(n-1)\Delta_t)}{\Delta_t}(u^n - u^{n-1}) + u^{n-1} \quad \text{for} \\ t \in [(n-1)\Delta_t, n\Delta_t),\ \forall\, n \in [1, \cdots, N], \\ f_{\Delta_t} = f^n \quad \text{for } t \in [(n-1)\Delta_t, n\Delta_t),\ \forall\, n \in [1, \cdots, N], \end{array}\right.$$

we have that

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} - \Delta_p u_{\Delta_t} - \frac{1}{{u_{\Delta_t}}^\delta} = f_{\Delta_t} \in L^\infty(Q_T). \tag{3}$$

We first establish some energy estimates independent of Δ_t. Multiplying (2) by u^n and summing from $n=1$ to N, we get

$$\Delta_t[\sum_{n=1}^{N}\int_\Omega \frac{u^n-u^{n-1}}{\Delta_t}u^n - \sum_{n=1}^{N} < \Delta_p u^n, u^n > - \sum_{n=1}^{N}\int_\Omega (u^n)^{1-\delta}] = \Delta_t\sum_{n=1}^{N}\int_\Omega f^n u^n.$$

Thus, for $\epsilon > 0$ small, by Young Inequality and Sobolev imbedding

$$\sum_{n=1}^{N}\int_\Omega (u^n-u^{n-1})u^n + \Delta_t[\sum_{n=1}^{N}\|u^n\|^p_{W_0^{1,p}(\Omega)} - \sum_{n=1}^{N}\int_\Omega (u^n)^{1-\delta}] \leq$$
$$\Delta_t\sum_{n=1}^{N}[C(\epsilon)\|f^n\|^{\frac{p}{p-1}}_{W^{-1,\frac{p}{p-1}}(\Omega)} + \epsilon\|u^n\|^p_{W_0^{1,p}(\Omega)}].$$

Moreover, by Jensen Inequality

$$\Delta_t\sum_{n=1}^{N}\|f^n\|^{\frac{p}{p-1}}_{W^{-1,\frac{p}{p-1}}(\Omega)} = \|f_{\Delta_t}\|^{\frac{p}{p-1}}_{L^{\frac{p}{p-1}}(0,T,W^{-1,\frac{p}{p-1}}(\Omega))} \leq \|f\|^{\frac{p}{p-1}}_{L^{\frac{p}{p-1}}(0,T,W^{-1,\frac{p}{p-1}}(\Omega))}$$
$$\leq C\|f\|^{\frac{p}{p-1}}_\infty.$$

In addition,

$$\sum_{n=1}^{N}\int_\Omega (u^n-u^{n-1})u^n = \tfrac{1}{2}\sum_{n=1}^{N}[\int_\Omega |u^n-u^{n-1}|^2 + |u^n|^2 - |u^{n-1}|^2]$$
$$= \tfrac{1}{2}\sum_{n=1}^{N}[\int_\Omega |u^n-u^{n-1}|^2 + \frac{1}{2}\int_\Omega |u^N|^2 - \frac{1}{2}\int_\Omega |u_0|^2.$$

Next, we estimate the singular term in the above expression. For that, arguing as in the proof of Theorem 1.3, we can prove the existence of $\underline{u}$, $\bar{u} \in W_0^{1,p}(\Omega)\cap\mathcal{C}$ such that

$$-\Delta_p\underline{u} - \frac{1}{\underline{u}^\delta} \leq -\|f\|_{L^\infty(Q_T)} \quad \text{in } \Omega,$$

$$-\Delta_p\bar{u} - \frac{1}{\bar{u}^\delta} \geq \|f\|_{L^\infty(Q_T)} \quad \text{in } \Omega,$$

and since $u_0 \in \mathcal{C}$, $\underline{u} \leq u_0 \leq \bar{u}$. Then from the weak comparison principle, by iteration, we obtain that $\underline{u} \leq u^n \leq \bar{u}$ which implies that

$$\underline{u} \leq u_{\Delta_t},\, \tilde{u}_{\Delta_t} \leq \bar{u}. \tag{4}$$

Therefore, since $\delta < 2 + \frac{1}{p-1}$

$$\Delta_t\sum_{n=1}^{N}\int_\Omega (u^n)^{1-\delta} \leq \begin{cases} T\int_\Omega \bar{u}^{1-\delta} < +\infty & \text{if } \delta \leq 1 \\ T\int_\Omega \underline{u}^{1-\delta} < +\infty & \text{if } \delta > 1. \end{cases}$$

Gathering the above estimates, we get that u_{Δ_t}, $\tilde{u}_{\Delta_t} \in \mathcal{C}$ uniformly and are bounded in $L^p(0,T;\, W_0^{1,p}(\Omega)) \cap L^\infty(0,T;\, L^2(\Omega))$. We now use a second energy estimate.

Multiplying (2) by $\frac{u^n-u^{n-1}}{\Delta_t}$ and summing from $n=1$ to N, we get

$$\Delta_t \sum_{n=1}^{N} \int_\Omega (\frac{u^n-u^{n-1}}{\Delta_t})^2 - \sum_{n=1}^{N} < \Delta_p u^n, u^n - u^{n-1} > - \sum_{n=1}^{N} \int_\Omega \frac{u^n-u^{n-1}}{u^{n\delta}} = \sum_{n=1}^{N} \int_\Omega f^n(u^n-u^{n-1}).$$

Thus, by the Young Inequality

$$\Delta_t \sum_{n=1}^{N} \int_\Omega (\frac{u^n-u^{n-1}}{\Delta_t})^2 + \sum_{n=1}^{N} \int_\Omega |\nabla u^n|^{p-2}\nabla u^n \cdot \nabla(u^n-u^{n-1}) - \sum_{n=1}^{N} \int_\Omega \frac{u^n-u^{n-1}}{u^{n\delta}} \leq \frac{\Delta_t}{2} \sum_{n=1}^{N} [\int_\Omega (f^n)^2 + \int_\Omega (\frac{u^n-u^{n-1}}{\Delta_t})^2]$$

which implies that

$$\tfrac{\Delta_t}{2} \sum_{n=1}^{N} \int_\Omega (\frac{u^n-u^{n-1}}{\Delta_t})^2 + \sum_{n=1}^{N} \int_\Omega |\nabla u^n|^{p-2}\nabla u^n \cdot \nabla(u^n-u^{n-1}) - \sum_{n=1}^{N} \int_\Omega \frac{u^n-u^{n-1}}{u^{n\delta}} \leq |\Omega| \frac{T}{2} \|f\|^2_{L^\infty(Q_T)}.$$

From the convexity of the terms $\int_\Omega |\nabla u|^p$ and $-\frac{1}{1-\delta}\int_\Omega u^{1-\delta}$ we derive the following estimates:

$$\frac{1}{p}[\int_\Omega |\nabla u^n|^p - \int_\Omega |\nabla u^{n-1}|^p] \leq \int_\Omega |\nabla u^n|^{p-2}\nabla u^n (\nabla(u^n-u^{n-1}),$$

$$\frac{1}{1-\delta}[\int_\Omega (u^{n-1})^{1-\delta} - \int_\Omega (u^n)^{1-\delta}] \leq -\int_\Omega \frac{u^n-u^{n-1}}{(u^n)^\delta}.$$

Therefore, gathering the above estimates, we get

$$\tfrac{\Delta_t}{2} \sum_{n=1}^{N} \int_\Omega (\frac{u^n-u^{n-1}}{\Delta_t})^2 + \frac{1}{p}[\int_\Omega |\nabla u^N|^p - \int_\Omega |\nabla u_0|^p] + \tfrac{1}{1-\delta}[\textstyle\int_\Omega (u_0)^{1-\delta} - \int_\Omega (u^N)^{1-\delta}] \leq |\Omega| \tfrac{T}{2} \|f\|^2_{L^\infty(Q_T)}.$$

Together with $\int_\Omega (u^n)^{1-\delta} \leq \max\{\int_\Omega (\bar{u})^{1-\delta}, \int_\Omega (\underline{u})^{1-\delta}\}$, it follows that $\frac{\partial \tilde{u}_{\Delta_t}}{\partial t}$ is bounded in $L^2(Q_T)$, u_{Δ_t}, $\tilde{u}_{\Delta_t}$ are bounded in $L^\infty(0,T;\, W_0^{1,p}(\Omega))$ uniformly in Δ_t. Furthermore, from above there exists $C > 0$ independent of Δ_t such that

$$\|u_{\Delta_t} - \tilde{u}_{\Delta_t}\|_{L^\infty(0,T;\, L^2(\Omega))} \leq \max_{n\in[1,\cdot\cdot,N]} \|u^n - u^{n-1}\|_{L^2(\Omega)} \leq C(\Delta_t)^{\frac{1}{2}}. \tag{5}$$

Therefore, taking $N \to \infty$ (which implies that $\Delta_t \to 0^+$), and up to a subsequence, we get from above estimates that there exists $u, v \in L^\infty(0,T;\, W_0^{1,p}(\Omega) \cap L^\infty(\Omega))$ such that $\frac{\partial u}{\partial t} \in L^2(Q_T)$, u, $v \in \mathcal{C}$ uniformly and as $\Delta_t \to 0^+$,

$$\tilde{u}_{\Delta_t} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;\, W_0^{1,p}(\Omega) \cap L^\infty(\Omega))$$

$$u_{\Delta_t} \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(0,T;\, W_0^{1,p}(\Omega) \cap L^\infty(\Omega))$$

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{in } L^2(Q_T).$$

From (5), it follows that $u \equiv v$. From (5), above estimates, from compactness Sobolev imbedding, from the interpolation inequality, and from Ascoli-Arzela Theorem, we get that

$$u_{\Delta_t},\, \tilde{u}_{\Delta_t} \to u \quad \text{in } L^\infty(0,T;\, L^q(\Omega)),\ \forall\, q > 1. \tag{6}$$

Now, multiplying (3) by $(u_{\Delta_t} - u)$ and using (6), we get after straightforward calculations:

$$\int_0^T \int_\Omega [\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} - \frac{\partial u}{\partial t}](\tilde{u}_{\Delta_t} - u) - \int_0^T < \Delta_p u_{\Delta_t}, u_{\Delta_t} - u > - \int_0^T \int_\Omega u_{\Delta_t}^{-\delta}(u_{\Delta_t} - u) = \int_0^T \int_\Omega f_{\Delta_t}(u_{\Delta_t} - u) + o_{\Delta_t}(1).$$

By Lebesgue Theorem and (4), we have that

$$\int_0^T \int_\Omega u_{\Delta_t}^{-\delta}(u_{\Delta_t} - u) = o_{\Delta_t}(1).$$

Up to a subsequence, we can assume that for some $q > 1$ $f_{\Delta_t} \rightharpoonup f$ in $L^{\frac{q}{q-1}}(Q_T)$ which implies together with (6) that

$$\int_0^T \int_\Omega f_{\Delta_t}(u_{\Delta_t} - u) = o_{\Delta_t}(1).$$

Then,

$$\frac{1}{2}\int_\Omega |\tilde{u}_{\Delta_t} - u|^2 - \int_0^T < \Delta_p u_{\Delta_t} - \Delta_p u, u_{\Delta_t} - u >= o_{\Delta_t}(1).$$

Therefore, using (6), $u \not\equiv 0$ and the following well-know inequality (for $v_1, v_2 \in W_0^{1,p},\Omega)$)

$$< -\Delta_p v_2 + \Delta_p v_1, v_2 - v_1 >\geq \begin{cases} C(\|v_2 - v_1\|^p_{W_0^{1,p}(\Omega)} & \text{if } p \geq 2 \\ C\dfrac{\|v_2 - v_1\|^2_{W_0^{1,p}(\Omega)}}{(\|v_2\|_{W_0^{1,p}(\Omega)} + \|v_1\|_{W_0^{1,p}(\Omega)})^{2-p}} & \text{if } p < 2 \end{cases}$$

with $C > 0$, we obtain that $u_{\Delta_t} \to u$ in $L^p(0,T;\, W_0^{1,p}(\Omega))$. Then,

$$-\Delta_p u_{\Delta_t} \to -\Delta_p u \quad \text{in } L^{\frac{p}{p-1}}(0,T;\, W^{-1,\frac{p}{p-1}}(\Omega)).$$

Moreover, from (4), for any $\phi \in W_0^{1,p}(\Omega)$

$$|\int_\Omega \frac{\phi}{(u_{\Delta_t})^\delta}| \leq \int_\Omega \frac{|\phi|}{(\underline{u})^\delta} \leq (\int_\Omega (\frac{d(x)}{(\underline{u})^\delta})^{\frac{p}{p-1}})^{\frac{p-1}{p}} \times (\int_\Omega (\frac{|\phi|}{d(x)})^p)^{\frac{1}{p}}$$

and since $\delta < 2 + \frac{1}{p-1}$

$$\int_\Omega (\frac{d(x)}{(\underline{u})^\delta})^{\frac{p}{p-1}} < +\infty.$$

Then, from the Hardy inequality and from Lebesgue Theorem, we obtain

$$\frac{1}{{u_{\Delta_t}}^\delta} \to \frac{1}{u^\delta} \quad \text{in } L^\infty(0,T;\, W^{-1,\frac{p}{p-1}}(\Omega)).$$

Therefore, $u \in \mathbf{V}(Q_T)$ satisfies (P$_t$). To complete the proof, let us show that u is the unique weak solution such that $u(t) \in \mathcal{C}$, $\forall\, t \in [0,T]$. Assume that there exists $v \not\equiv u$ a weak solution to (P$_t$) satisfying $v(t) \in \mathcal{C}$, $\forall\, t \in [0,T]$. Then,

$$\int_0^T \int_\Omega \frac{\partial(u-v)}{\partial t}(u - v) - \int_0^T < \Delta_p u + \Delta_p v, u - v > - \int_0^T \int_\Omega (u^{-\delta} - v^{-\delta})(u - v) = 0$$

together with $u(0) = v(0)$ implies $u \equiv v$. □

REFERENCES

[1] A. Anane, *Simplicité et isolation de la première valeur propre du p-laplacien avec poids*, Comptes Rendus Acad. Sc. Paris, Série I, Math., **305** (1987), 725-728.

[2] A. Anane, *"Etude des valeurs propres et de la résonance pour l'opérateur p-Laplacien"*, Thèse de doctorat, Université Libre de Bruxelles, 1988, Brussels.

[3] C. Aranda and T. Godoy, *Existence and multiplicity of positive solutions for a singular problem associated to the p-Laplacian operator*, Electron. J. Differential Equations, **132** (2004), 1-15.

[4] E. DiBenedetto, $C^{1+\alpha}$ *local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal., **7**(8) (1983), 827-850.

[5] M. G. Crandall, P. H. Rabinowitz, and L. Tartar, *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations, **2** (1977), 193-222.

[6] M. Cuesta and P. Takáč, *A strong comparison principle for positive solutions of degenerate elliptic equations*, Differential and Integral Equations, **13**(4-6) (2000), 721-746.

[7] J. Davila and M. Montenegro, *Existence and asymptotic behavior for a singular parabolic equation*, Trans. Amer. Math. Soc. **357** no. 5 (2005), 1801-1828.

[8] K. Deimling, *"Nonlinear functional analysis"*, Springer-Verlag Berlin–Heidelberg–New York, 1985.

[9] L. Dupaigne, M. Ghergu and V. Radulescu, *Lane-Emden-Fowler equations with convection and singular potential*, J. Math. Pures et Appliquées **87** (2007), 563-581.

[10] J. Fleckinger and P. Takáč, *Uniqueness of positive solutions for nonlinear cooperative systems with the p-laplacian*, Indiana Univ. math. J., **43** (4) (1994), 1227-1253.

[11] M. Ghergu and V. Radulescu, *Singular elliptic problems. Bifurcation and asymptotic analysis.* Oxford University Press, 2008.

[12] J. Giacomoni, I. Schindler and Peter Takáč, *Sobolev versus Hölder local minimizers and existence of multiple solutions foa a singular quasilinear equation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) **6** (2007), 117-158.

[13] J. Hernández and F. J. Mancebo, Singular Elliptic and Parabolic Equations, Handbook of Differential Equations Vol. 3 (2006), 317-400.

[14] J. Hernández, F. Mancebo, and J. M. Vega, *On the linearization of some singular, nonlinear elliptic problems and applications*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **19**(6) (2002), 777-813.

[15] K. Perera and E. A. B. Silva, *On singular p-Laplacian problems*, Diff. Int. equations **20** (2007), 105-120.

[16] J. B. Serrin, *Local behavior of solutions of quasilinear elliptic equations*, Acta Math., **111** (1964), 247-302.

[17] P. Tolksdorf, *On the Dirichlet problem for quasilinear equations in domains with conical boundary points*, Comm. Partial Diff Equations, **8**(7) (1983), 773-817.

[18] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations, **51** (1984), 126-150.

[19] M. Winkler, *Nonuniqueness in the quenching problem*, Math. Ann., **339** (2007), 559-597.

Received July 2010; revised January 2011.

E-mail address: mehdi.badra@univ-pau.fr
E-mail address: kausbal@gmail.com
E-mail address: jgiacomo@univ-pau.fr