SOME RESULTS ABOUT A QUASILINEAR SINGULAR PARABOLIC EQUATION

MEHDI BADRA, KAUSHIK BAL AND JACQUES GIACOMONI

Dedicated to Professor Jesús Ildefonso Díaz
on the occasion of his 60th birthday

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Abstract. We investigate the following quasilinear parabolic and singular equation,

$$\begin{cases} u_t - \Delta_p u = \dfrac{1}{u^\delta} + f(x,u) \text{ in } (0,T)\times\Omega, \\ u = 0 \text{ on } (0,T)\times\partial\Omega, \quad u > 0 \text{ in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) \text{ in } \Omega, \end{cases} \tag{P_t}$$

where Ω is an open bounded domain with smooth boundary in $\mathbb{R}^{\mathrm{N}}$, $1 < p < \infty$, $0 < \delta$ and $T > 0$. We assume that $(x,s) \in \Omega\times\mathbb{R}^+ \to f(x,s)$ is a bounded below Caratheodory function, asymptotically sub-homogeneous, i.e.

$$\begin{cases} \text{if } p \leqslant 2,\ 0 \leqslant \limsup\limits_{t\to+\infty} \dfrac{f(x,t)}{t^{p-1}} = \alpha_f < \lambda_1(\Omega), \\ \text{if } p > 2,\ 0 \leqslant \limsup\limits_{t\to+\infty} \dfrac{f(x,t)}{t} = \alpha_f < \infty, \end{cases} \tag{0.1}$$

(where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta_p$ in Ω with homogeneous Dirichlet boundary conditions) and $u_0 \in W_0^{1,p}(\Omega)$. Then, for any $\delta \in (0,1)$, we prove for any $T > 0$ the existence of a weak solution $u \in \mathbf{V}(Q_T)$ to $(\mathrm{P_t})$. The proof involves a semi-discretization in time approach and the study of the stationary problem associated to $(\mathrm{P_t})$. The key points in the proof is to show that the approximated solutions remain (uniformly) positive in any compact K of Ω and from energy estimates converges to a weak solution to $(\mathrm{P_t})$. Next, under additional assumptions on the initial data, δ and the nonlinearity f, we prove long time convergence of global weak solutions in $W_0^{1,p}(\Omega)$. This stabilization property is established by proving an additional energy estimate and by using the regularity result in Simon [23]. These results extend with a different approach a previous work of the authors ([3]) regarding the problem $(\mathrm{P_t})$ where existence and uniqueness of solutions are proved under a cone condition on the initial data and via the theory of nonlinear accretive operators.

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1. Introduction

In the present paper we investigate the following quasilinear and singular parabolic problem :

$$\begin{cases} u_t - \Delta_p u = \dfrac{1}{u^\delta} + f(x,u) \ \text{ in } Q_T, \\ u = 0 \text{ on } \Sigma_T, \quad u > 0 \text{ in } Q_T, \\ u(0,x) = u_0(x) \ \text{ in } \Omega, \end{cases} \tag{P$_t$}$$

where Ω is an open bounded domain with smooth boundary in $\mathbb{R}^{\mathrm{N}}$ (with $\mathrm{N} \geqslant 2$), $1 < p < \infty$, $0 < \delta$, $T > 0$, $Q_T = (0,T) \times \Omega$ and $\Sigma_T = (0,T) \times \partial\Omega$. We assume that f is a bounded below Caratheodory function and satisfying (0.1) and $u_0 \in W_0^{1,p}(\Omega)$. Such a problem arises in the study of non-Newtonian fluids (in particular *pseudoplastic* fluids), boundary-layer phenomena for viscous fluids (see [9], [17], [18]), in the Langmuir-Hinshelwood model of chemical hetereogeneous catalyst kinetics (see [2], [21]), in enzymatic kinetics models (see [4]), as well as in the theory of heat conduction in electrically conducting materials (see [16]) and in the study of guided modes of an electromagnetic field in nonlinear medium (see [11]). Problem (P_t) with $p \neq 2$ arises specifically in the study of turbulent flow of a gas in porous media (see [19]). We refer to the survey Hernández-Mancebo-Vega [15], the book Ghergu-Radulescu [12] and the bibliography therein for more details about the corresponding models.

We are particularly interested to discuss existence of weak solutions and the behaviour of global weak solutions. The notion of weak solutions is stated below: First, we introduce

DEFINITION 1.1. $\mathbf{V}(Q_T) \stackrel{\text{def}}{=} \left\{u \in L^2(Q_T) : u \in L^\infty(0,T; W_0^{1,p}(\Omega)), u_t \in L^2(Q_T)\right\}$.

Then, we define

DEFINITION 1.2. A *weak* solution to (P_t) is a function $u \in \mathbf{V}(Q_T)$ satisfying:

1. for any compact $K \subset [0,T] \times \Omega$, $\operatorname*{ess\,inf}_K u > 0$,
2. for every test function $\phi \in C_c^\infty([0,T] \times \Omega)$,

$$\int_{Q_T} \Big(\phi \frac{\partial u}{\partial t} + |\nabla u|^{p-2} \nabla u \nabla \phi - \phi(\frac{1}{u^\delta} + f(t,u))\Big) \mathrm{d}x\mathrm{d}t = 0, \tag{1.1}$$

3. $u(0,x) = u_0(x)$ a.e in Ω.

To prove existence of weak solutions, we use a semi-discretization in time. Precisely, taking advantage of the study of an auxiliary quasilinear and singular elliptic equation and energy estimates, we are able to prove existence of weak solutions to (P_t) for $u_0 \in W_0^{1,p}(\Omega)$ and positive in Ω. We state this result below:

THEOREM 1.3. *Let* $T > 0$, $p > 2N/(N+2)$, $0 < \delta < 1$ *and* $u_0 \in W_0^{1,p}(\Omega)$ *such that for any compact* $K \subset \Omega$, $\operatorname*{ess\,inf}_K u_0 > 0$. *Assume that* f *is a bounded below caratheodory function, satisfying* (0.1). *Then, there exists a weak solution to* (P_t).

We stress that this result does not require any control of u_0 near the boundary in contrast with [Theorem 1.4, [3]] and existence results in [15] and in [24]. In our knowledge, Theorem 1.3 is the first result (even for $p=2$) showing existence of (very) weak solutions to this class of quasilinear singular parabolic equations with intial data only positive in Ω. Regarding Theorem 1.4, the proof of Theorem 1.3 follows a different approach based on convexity arguments (see the beginning of the proof of Theorem 1.3 pages 6-7) whereas the proof of Theorem 1.4 uses strongly the construction of suitable "uniform" sub- and supersolutions which control the singular term along the flow. We emphasize that the existence of such sub- and supersolutions is guaranteed by u_0 in the conical shell $\mathscr{C}$ defined below. Consequently, to establish Theorem 1.3, we have to face an important difficulty, that is to show that the discrete solutions u_{Δ_t}, $\tilde{u}_{\Delta_t}$ are well defined and satisfy (2.5) in the sense of distributions.

Next, we investigate the long time behaviour of weak solutions. In this regard, we recall some results proved for problem (P_t) in [3] where under a cone condition which prescribes the behaviour of the initial data near the boundary, the existence and the uniqueness of a weak solution is proved. Precisely, let $\mathscr{C}$ be the set of functions $v \in L^\infty(\Omega)$ such that there exists $c_1>0$ and $c_2 \geqslant c_1$ satisfying

$$\begin{cases} c_1 d(x) \leqslant v \leqslant c_2 d(x) & \text{if } \delta<1, \\ c_1 d(x)\log^{\frac{1}{p}}\Big(\dfrac{k}{d(x)}\Big) \leqslant v \leqslant c_2 d(x)\log^{\frac{1}{p}}\Big(\dfrac{k}{d(x)}\Big) & \text{if } \delta=1, \\ c_1 d(x)^{\frac{p}{\delta+p-1}} \leqslant v \leqslant c_2\big(d(x)^{\frac{p}{\delta+p-1}}+d(x)\big) & \text{if } \delta>1, \end{cases}$$

where $d(x) \stackrel{\text{def}}{=} \operatorname{dist}(x,\partial\Omega)$ and $k>0$ is large enough. Then, we have

THEOREM 1.4. (Badra-Bal-Giacomoni) [3] *Let $0<\delta<2+1/(p-1)$. Assume that f is a bounded below Caratheodory function, and that f is locally Lipschitz with respect to the second variable uniformly in $x\in\Omega$ and satisfying*

$$0 \leqslant \limsup_{t\to+\infty} \frac{f(x,t)}{t^{p-1}} < \lambda_1(\Omega).$$

Let $u_0 \in W_0^{1,p}(\Omega)\cap\mathscr{C}$. Then, for any $T>0$, there exists a unique weak solution, u, to $(\mathrm{P_t})$ such that $u(t)\in\mathscr{C}$ uniformly for $t\in[0,T]$, $u\in C([0,T],W_0^{1,p}(\Omega))$ and u satisfies for any $t\in[0,T]$:

$$\begin{aligned} \int_0^t\int_\Omega \Big(\frac{\partial u}{\partial t}\Big)^2 \mathrm{d}x\mathrm{d}s + \frac{1}{p}\int_\Omega |\nabla u(t)|^p \mathrm{d}x - \frac{1}{1-\delta}\int_\Omega u^{1-\delta}(t)\mathrm{d}x \\ = \int_\Omega F(x,u(t))\mathrm{d}x + \frac{1}{p}\int_\Omega |\nabla u_0|^p\mathrm{d}x \\ - \frac{1}{1-\delta}\int_\Omega u_0^{1-\delta}\mathrm{d}x - \int_\Omega F(x,u_0)\mathrm{d}x, \quad (1.2) \end{aligned}$$

where $F(x,w) \stackrel{\text{def}}{=} \int_0^w f(x,s)\mathrm{d}s$.

The control of the singular term is given by the cone condition and the following Hardy Inequality (see for instance [6, chapter 9]):

THEOREM 1.5. *Let Ω be a bounded open set of class C^1 and let $1 < p < \infty$. There exists a constant $C > 0$ such that*

$$\left\|\frac{u}{d}\right\|_{L^p(\Omega)} \leqslant C\|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega).$$

Conversely,

$$u \in W^{1,p}(\Omega) \text{ and } (u/d) \in L^p(\Omega) \Rightarrow u \in W_0^{1,p}(\Omega).$$

The restriction $\delta < 2+1/(p-1)$ is optimal since it can be proved that for δ beyond $2+1/(p-1)$ stationary solutions do not belong to $W_0^{1,p}(\Omega)$. Notice that the solution given by Theorem 1.4, namely u, satisfies

$$\frac{1}{u^\delta} \in L^\infty(0,T;W_0^{-1,p'}(\Omega)) \quad \text{with} \quad p' \stackrel{\text{def}}{=} \frac{p}{p-1}$$

and then (1.1) is verified by any $\phi \in \mathbf{V}(Q_T)$.

Under additional conditions on f, the uniqueness of the stationary solution can be derived:

THEOREM 1.6. [3] *Let $0 < \delta < 2+1/(p-1)$ and $f : \Omega\times\mathbb{R}^+ \to \mathbb{R}$ be a bounded below Caratheodory function, locally Lipschitz with respect to the second variable uniformly in $x \in \Omega$, satisfying*

$$0 \leqslant \limsup_{t\to+\infty} \frac{f(x,t)}{t^{p-1}} < \lambda_1(\Omega)$$

and such that $f(x,s)/s^{p-1}$ is a decreasing function in $\mathbb{R}^+$ for a.e. $x \in \Omega$. Then there exists a unique u_∞ in $W_0^{1,p}(\Omega)\cap\mathscr{C}\cap C_0(\overline{\Omega})$ satisfying

$$(\mathrm{Q})\begin{cases} -\Delta_p u_\infty - \dfrac{1}{u_\infty^\delta} = f(x,u_\infty) & \text{in } \Omega, \\ u_\infty = 0 & \text{on } \partial\Omega. \end{cases}$$

The proof of the above theorem given in [3] uses strongly the Díaz-Saa inequality (see [10]). From m-accretivity of the operator A defined by $Au \stackrel{\text{def}}{=} -\Delta_p u - \frac{1}{u^\delta}$ in $L^\infty(\Omega)$ and from Theorem 1.6, it follows that

THEOREM 1.7. [3] *Let hypothesis in Theorem 1.4 satisfied and assume that*

$$\frac{f(x,s)}{s^{p-1}} \quad \textit{is decreasing in } (0,\infty) \textit{ for a.e. } x \in \Omega.$$

Then, the solution to (P_t) is defined in $(0,\infty)\times\Omega$ and satisfies

$$u(t) \to u_\infty \quad \text{in } L^\infty(\Omega) \quad \text{as } t\to\infty, \tag{1.3}$$

where u_∞ is defined in Theorem 1.6.

From maximal regularity L^p-L^q results for the heat equation, Sobolev interpolation theory and Hardy type inequalities (see [25, Par. 3.2.6, Lem. 3.2.6.1, p.259]) of the form given by:

THEOREM 1.8. *Let $s \in [0,2]$ such that $s \neq 1/2$ and $s \neq 3/2$. Then the following generalisation of Hardy's inequality holds:*

$$\|d^{-s}g\|_{L^2(\Omega)} \leqslant C\|g\|_{H^s(\Omega)} \quad \text{for all } g \in H_0^s(\Omega); \tag{1.4}$$

we get in the case $p = 2$ stabilization in the energy space $H_0^1(\Omega)$. Precisely, one has

THEOREM 1.9. [3] *Let $p = 2$, $\delta < 3$, $u_0 \in \mathscr{C} \cap H_0^1(\Omega)$. Assume that f satisfies the hypothesis in Theorem* 1.7. *Then, the solution to* (P_t), *u, defined in* $(0,\infty)\times\Omega$ *satisfies the following asymptotic behaviour: $u(t) \to u_\infty$ as $t \to +\infty$ in $L^\infty(\Omega) \cap H_0^1(\Omega)$.*

In our present paper, we further consider stabilization properties of solutions to (P_t) and we show the following result:

THEOREM 1.10. *Let conditions in Theorem* 1.7 *be satisfied and assume either $p > 2$ and $\delta < (p-1)/p$ or the following conditions $N \leqslant 3$, $N/2 \leqslant p < 2$ and $\delta < 1/2$. Then,*

$$u(t) \to u_\infty \quad \text{in } L^\infty(\Omega) \cap W_0^{1,p}(\Omega) \quad \text{as } t \to \infty. \tag{1.5}$$

According to our knowledge, Theorem 1.10 is the first result which gives the asymptotic convergence of solutions to (P_t) to a stationary solution in $W_0^{1,p}(\Omega)$ for $p \neq 2$. Regarding Theorem 1.7 where stabilization is proved in $L^\infty(\Omega)$, the proof of Theorem 1.10 relies on a different approach based on a regularity result in Besov spaces implying the compactness of trajectories in $W_0^{1,p}(\Omega)$ whereas the proof of Theorem 1.7 uses fully the maximal accretivity of A in $L^\infty(\Omega)$ that gives the continuity and monotonicity (for suitable initial data) of trajectories. To establish Theorem 1.10, we then need to prove additional energy estimates on the approximated solutions compared to [3]. Precisely, in the present paper, we establish the following new result:

PROPOSITION 1.11. *Let conditions in Theorem* 1.4 *be satisfied. Then, for any $t_0 > 0$, the solution u verifies:*

$$\frac{\partial u}{\partial t} \in L^2(]0,\infty[, L^2(\Omega)) \cap L^\infty([t_0,+\infty[, L^2(\Omega)).$$

We stress that Proposition 1.11 holds for δ in $(0, 2+1/(p-1))$. We combine proposition 1.11 with regularity results for p-Laplace equations in Besov spaces due to J. Simon [23] (see Theorem 3.3) to get compactness of solutions to (P_t) for large time t in $W_0^{1,p}(\Omega)$. Precisely, we show the following new result:

PROPOSITION 1.12. *Let the conditions in Theorem* 1.4 *be satisfied. Then for any* $t_0 > 0$*, the solution* u *to* (P_t) *verifies:*
i) *If* $p > 2$ *and* $\delta < (p-1)/p$*, then for any* $t_0 > 0$*,*

$$u \in L^\infty([t_0,+\infty), B_\infty^{1+\frac{1}{(p-1)^2},p}(\Omega)),$$

where

$$B_\infty^{1+\frac{1}{(p-1)^2},p} \stackrel{\text{def}}{=} [W^{2,p}(\Omega), W^{1,p}(\Omega)]_{1-\frac{1}{(p-1)^2},\infty}$$

is the Besov space obtained by the real interpolation method.
ii) *If* $N \leqslant 3$*,* $\delta < 1/2$ *and* $N/2 \leqslant p < 2$*, then*

$$u \in L^\infty([t_0,+\infty), B_\infty^{1+(p-1)^2,p}(\Omega)),$$

where

$$B_\infty^{1+(p-1)^2,p}(\Omega) \stackrel{\text{def}}{=} [W^{2,p}(\Omega), W^{1,p}(\Omega)]_{1-(p-1)^2,\infty}.$$

We now give briefly the state of art concerning parabolic quasilinear singular equations. The corresponding stationary equation was studied intensively in the litterature. In particular the case $p = 2$, mostly when $\delta < 1$ and under different assumptions on the asymptotic behaviour of f was considered in detail (see the pionniering work Crandall-Rabinowitz-Tartar [7], the bibliography in Hernández-Mancebo [14]). The quasilinear case, namely $p \neq 2$, was not considered so far. We mention the work Aranda-Godoy [1] where existence results are obtained via the bifurcation theory for $1 < p \leqslant 2$ and $f(x,u) = g(u)$ satisfying some growth conditions. In Giacomoni-Schindler-Takáč [13] the existence and multiplicity results when $1 < p < \infty$ $f(x,u) = u^q$ with $p-1 < q \leqslant p^* - 1$ and $0 < \delta < 1$ are proved by using variational methods and regularity results in Hölder spaces. In Perera-Silva [20], other kinds of singularities are investigated (for instance $e^{\frac{1}{u}}$ instead of $1/u^\delta$). In Boccardo-Orsina [5], nonexistence results are proved for quasilinear equations involving singular terms in the form $q(x)/u^\delta$ where q belongs to a certain class of bounded Radon measure (for instance a Dirac mass). Concerning the parabolic case, avalaible results mostly concern the case $p = 2$. Namely, we first quote the result in Hernandez-Mancebo-Vega [15] where properties of the linearised operator (in $C_0^1(\overline{\Omega})$) and the validity of the strong maximum principle are given, that induce the asymptotic stability of a certain class of stationary solutions in the range $0 < \delta < \frac{1}{2}$. In Takáč [24], a stabilization result in C^1 is proved for a similar class of parabolic singular problems via a clever use of weighted Sobolev spaces. Notice that the common feature of these two works is that solutions belong to $\left[C_0^1(\overline{\Omega})\right]^+$, the interior of the positive cone of $C_0^1(\overline{\Omega})$, that gives an implicit control of the singular terms near the boundary $\partial\Omega$. However, in the context of Problem (P_t) this approach fails for large δ (that is for $\delta \geqslant 1$) since weak solutions do not belong to $C^1(\overline{\Omega})$. In [3], we overcome this difficulty by showing the invariance of a conical shell, namely $\mathscr{C}$, along the flow associated to (P_t). In the present paper (Theorem 1.3), we show that this cone condition can be removed when $\delta < 1$. However, we obtain weaker solutions in this case.

We also mention the work Davila-Montenegro [8] still concerning the case $p=2$ and with singular absorption term. In this nice work, the authors achieved uniqueness within the class of functions satisfying $u(x,t) \geqslant c\,\mathrm{dist}(x,\partial\Omega)^{\gamma}$ for suitable γ and $c>0$ and discuss the asymptotic behaviour of solutions. Finally, we would like to quote the nice paper Winkler [26], where the author shows that uniqueness is violated in case of non homogeneous boundary Dirichlet condition.

Our present paper is organized as follows. In the next section (Section 2), we prove Theorem 1.3. In Section 3, we focus on stabilization of solutions to (P_t) in $W_0^{1,p}(\Omega)$ and give the proof of propositions 1.11 and 1.12 and finally Theorem 1.10.

2. Proof of Theorem 1.3

In this section, we prove Theorem 1.3.

Proof. Consider $\phi_{1,K}$ the normalised positive eigenfunction associated to $\lambda_{1,K}$, the first eigenvalue of $-\Delta_p$ in (the interior of) a compact and smooth set K $(\subset \Omega)$ with Dirichlet boundary Conditions. We have for $\eta = \eta(K) > 0$ small enough that $u_0 \geqslant \eta\phi_{1,K}$ in K and

$$\frac{\eta\phi_{1,K} - u_0}{\Delta_t} - \Delta_p \eta\phi_{1,K} - \frac{1}{(\eta\phi_{1,K})^{\delta}} - f(x,u_0) < 0 \text{ in } K.$$

We use the following iterative scheme to define approximated solutions, namely u_{Δ_t} and $\tilde{u}_{\Delta_t}$. More specifically, let $N \in \mathbb{N}\backslash\{0\}$ and denote $\Delta_t \stackrel{\mathrm{def}}{=} T/N$ and for any $t \in \mathbb{R}$, $t^+ \stackrel{\mathrm{def}}{=} \max\{t,0\}$. We construct a sequence $(u^n)_{n\in\mathbb{N}\backslash\{0\}} \subset W_0^{1,p}(\Omega)$, verifying

$$u^n - \Delta_t\left(\Delta_p u^n + \frac{1}{(u^n)^{\delta}}\right) = \Delta_t f(x,u^{n-1}) + u^{n-1} \quad \text{in } \Omega, \tag{2.1}$$

and we define $u^0 \stackrel{\mathrm{def}}{=} u_0 \in W_0^{1,p}(\Omega)$. Let us show the existence of u^n for any $n \in \mathbb{N}\backslash\{0\}$ satisfying (2.1) that means that for any $\phi \in W_0^{1,p}(\Omega)$,

$$\int_{\Omega}\Big(\phi(\frac{u^n - u^{n-1}}{\Delta_t}) + |\nabla u^n|^{p-2}\nabla u^n \nabla\phi - \phi(\frac{1}{u^{n\delta}} + f(x,u^{n-1}))\Big)\mathrm{d}x = 0. \tag{2.2}$$

For that, assuming that

$$u^{n-1} \in W_0^{1,p}(\Omega) \quad \text{and} \quad u^{n-1} \geqslant \eta\phi_{1,K} \text{ on } K,$$

we consider the following energy functional E_n defined by

$$E_n(u) \stackrel{\mathrm{def}}{=} \frac{1}{\Delta_t}\left(\int_{\Omega}\frac{u^2}{2}\mathrm{d}x - \int_{\Omega} u u^{n-1}\mathrm{d}x\right) + \frac{1}{p}\int_{\Omega}|\nabla u|^p\mathrm{d}x - \int_{\Omega}\frac{(u^+)^{1-\delta}}{1-\delta}\mathrm{d}x - \int_{\Omega} f(x,u^{n-1})u^+\mathrm{d}x,$$

for any $u \in W_0^{1,p}(\Omega)$. Notice that from Sobolev imbeddings and since f is bounded below and satisfies (0.1) (that both imply that $f(x,u^{n-1}) \in L^2(\Omega)$), E_n is bounded by below, coercive, weakly lower semicontinuous in $W_0^{1,p}(\Omega)$ and strictly convex in the positive cone of $W_0^{1,p}(\Omega)$. Then, from the fact that $E_n(u^+) \leqslant E_n(u)$ for any $u \in W_0^{1,p}(\Omega)$, E_n admits a unique global minimizer, we denote by u^n, in $W_0^{1,p}(\Omega)$ and $u^n \geqslant 0$ a.e. in Ω. We now prove that

$$u^n \geqslant \eta\phi_{1,K} \quad \text{on } K.$$

For that, let us consider

$$\psi_k = (\eta\phi_{1,K} - u^n)^+ \in W_0^{1,p}(\Omega)$$

after extending $\phi_{1,K}$ by 0 in $\Omega\backslash K$. Let us notice that ψ_k has a compact support included in K. Since u^n is the global minimizer of E_n,

$$\xi(t) = E_n(u^n + t\psi_k) \geqslant E(u^n)\ \forall t \geqslant 0.$$

Moreover, since $\phi_{1,K}$ satisfies $\phi_{1,K} \geqslant \eta_0 \mathrm{dist}(x,\partial K)$ for some $\eta_0 > 0$ small enough and $\delta < 1$, we have by Lebesgue theorem that

$$\lim_{t\to t_0} \frac{1}{t-t_0}\left[\int_\Omega \frac{(u^n+t\psi_k)^{1-\delta}}{1-\delta}\mathrm{d}x - \int_\Omega \frac{(u^n+t_0\psi_k)^{1-\delta}}{1-\delta}\mathrm{d}x\right] = \int_K (u^n+t_0\psi_k)^{-\delta}\psi_k \mathrm{d}x$$

for $t_0 \in (0,1]$ and then ξ is differentiable in $(0,1]$. From the convexity of ξ and the variational nature of u^n, we obtain that

$$\forall t \in (0,1),\ 0 \leqslant \xi'(t) \leqslant \xi'(1). \tag{2.3}$$

Furthermore,

$$\xi'(1) = \left\langle \frac{\eta\phi_{1,K} - u^{n-1}}{\Delta_t} - \Delta_p(\eta\phi_{1,K}) - \frac{1}{(\eta\phi_{1,K})^\delta} - f(x,u^{n-1}), \psi_k \right\rangle.$$

If the measure of the support of ψ_k is different from zero, we get that $\xi'(1) < 0$ and thereby a contradiction with (2.3). Thus,

$$\eta\phi_{1,K} \leqslant u^n \ \text{ in } K \text{ for every } n \geqslant 0.$$

Then, for $\phi \in C_c^\infty(\Omega)$,

$$\lim_{t\to 0} \frac{E_n(u^n+t\phi) - E_n(u^n)}{t} = 0.$$

Consequently, u^n satisfies the Euler-Lagrange equation, namely (2.1), in the sense of distributions, that is (2.2) is satisfied for any $\phi \in C_c^\infty(\Omega)$. By a density argument and since $u^n \in W_0^{1,p}(\Omega)$, we get that (2.2) is satisfied for any $\phi \in W_0^{1,p}(\Omega)$.

So consequently, $u_{\Delta_t}, \tilde{u}_{\Delta_t}$ set by: for all $n \in \{1, \dots, N\}$,

$$\forall t \in [(n-1)\Delta_t, n\Delta_t), \begin{cases} u_{\Delta_t}(t) \stackrel{\text{def}}{=} u^n, \\ \tilde{u}_{\Delta_t}(t) \stackrel{\text{def}}{=} \dfrac{(t-(n-1)\Delta_t)}{\Delta_t}(u^n - u^{n-1}) + u^{n-1}, \end{cases} \tag{2.4}$$

are well defined and satisfied in addition $u_{\Delta_t}, \tilde{u}_{\Delta_t} \geqslant \eta \phi_{1,K}$ on each compact K of Ω. In addition, we have that

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} - \Delta_p u_{\Delta_t} - \frac{1}{{u_{\Delta_t}}^{\delta}} = f(x, u_{\Delta_t}(\cdot - \Delta_t)) \tag{2.5}$$

which implies that for any $\phi \in C_c^\infty([0,T] \times \Omega)$

$$\int_0^T \int_\Omega \frac{\partial \tilde{u}_{\Delta_t}}{\partial t} \phi \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega |\nabla u_{\Delta_t}|^{p-2} \nabla u_{\Delta_t} \nabla \phi \mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega \frac{1}{{u_{\Delta_t}}^{\delta}} \phi \mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \int_\Omega f(x, u_{\Delta_t}(\cdot - \Delta_t)) \phi \mathrm{d}x\mathrm{d}t. \tag{2.6}$$

We have that

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} \in L^\infty(0,T; W_0^{1,p}(\Omega))$$

and from $p > 2N/(N+2)$ and (0.1), we obtain that

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t}, \ \ \Delta_p u_{\Delta_t}, \ \ f(x, u_{\Delta_t}(\cdot - \Delta_t)) \quad \text{belong to } L^{p'}(0,T; W^{-1,p'}(\Omega)).$$

Then, (2.5) holds in $(L^p(0,T,W_0^{1,p}(\Omega)))'$. Therefore, for any $\phi \in L^p(0,T,W_0^{1,p}(\Omega))$,

$$\int_0^T \int_\Omega \frac{\partial \tilde{u}_{\Delta_t}}{\partial t} \phi \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega |\nabla u_{\Delta_t}|^{p-2} \nabla u_{\Delta_t} \nabla \phi \mathrm{d}x\mathrm{d}t$$
$$- \int_0^T \int_\Omega \frac{1}{{u_{\Delta_t}}^{\delta}} \phi \mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \int_\Omega f(x, u_{\Delta_t}(\cdot - \Delta_t)) \phi \mathrm{d}x\mathrm{d}t. \tag{2.7}$$

We now derive some energy estimates on $u_{\Delta_t}, \tilde{u}_{\Delta_t}$.

Multiplying (2.1) by $\Delta_t u^n$, summing from $n = 1$ to N and integrating over Ω we obtain

$$\Delta_t \sum_{n=1}^N \int_\Omega \frac{u^n - u^{n-1}}{\Delta_t} u^n \mathrm{d}x + \Delta_t \sum_{n=1}^N \int_\Omega |\nabla u^n|^p \mathrm{d}x - \Delta_t \sum_{n=1}^N \int_\Omega {u^n}^{1-\delta} \mathrm{d}x$$
$$= \Delta_t \sum_{n=1}^N \int_\Omega f(x, u^{n-1}) u^n \mathrm{d}x.$$

The above expression implies

$$\sum_{n=1}^{N}\frac{1}{2}\int_{\Omega}(|u^n|^2-|u^{n-1}|^2+|u^n-u^{n-1}|^2)\mathrm{d}x$$
$$+\Delta_t\sum_{n=1}^{N}\int_{\Omega}|\nabla u^n|^p\mathrm{d}x-\Delta t\sum_{n=1}^{N}\int_{\Omega}(u^n)^{1-\delta}\mathrm{d}x$$
$$=\Delta_t\sum_{n=1}^{N}\int_{\Omega}f(x,u^{n-1})u^n\mathrm{d}x.$$

Now since f is bounded by below and satisfies (0.1), $\exists\, C=C(\alpha)>0$ large enough such that

$$f(x,t)\leqslant \alpha t^{p-1}+C, \tag{2.8}$$

where $\alpha<\lambda_1(\Omega)$.

Then, the term $\Delta_t\sum_{n=1}^{N}\int_{\Omega}f(x,u^{n-1})u^n\mathrm{d}x$ in the right hand side can be estimated as follows:

$$\Delta_t\sum_{n=1}^{N}\int_{\Omega}f(x,u^{n-1})u^n\mathrm{d}x\leqslant \Delta_t\sum_{n=1}^{N}[\alpha\int_{\Omega}(u^{n-1})^{p-1}u^n\mathrm{d}x+C\int_{\Omega}u^n\mathrm{d}x].$$

Now, applying Young's Inequality on the two last terms in the right-hand side we obtain for any $\varepsilon>0$

$$\Delta_t\sum_{n=1}^{N}\int_{\Omega}f(x,u^{n-1})u^n\mathrm{d}x\leqslant \alpha\Delta_t\sum_{n=1}^{N}\left[\frac{p-1}{p}\int_{\Omega}(u^{n-1})^p\mathrm{d}x+\frac{1}{p}\int_{\Omega}(u^n)^p\mathrm{d}x\right]$$
$$+\Delta_t\varepsilon\sum_{n=1}^{N}\int_{\Omega}(u^n)^p\mathrm{d}x+C_\varepsilon T|\Omega|.$$

for a constant $C_\varepsilon>0$ large enough depending only on ε and α.

Using again the Young inequality, we estimate the last term in the left-hand side in the following way

$$\Delta_t\sum_{n=1}^{N}\int_{\Omega}(u^n)^{1-\delta}\mathrm{d}x\leqslant \Delta_t\varepsilon\sum_{n=1}^{N}\int_{\Omega}(u^n)^p\mathrm{d}x+C_\varepsilon{}'T|\Omega|,$$

where $C'_\varepsilon>0$ is large enough and depends only on ε and δ.

Now using Poincaré inequality we get,

$$\frac{1}{2}\sum_{n=1}^{N}\int_{\Omega}(|u^n|^2-|u^{n-1}|^2+|u^n-u^{n-1}|^2)\mathrm{d}x+\Delta_t\Big(1-\frac{\alpha+2\varepsilon}{\lambda_1(\Omega)}\Big)\sum_{n=1}^{N}\int_{\Omega}|\nabla u^n|^p\mathrm{d}x$$
$$\leqslant \tilde{C}|\Omega|T+\alpha\Delta_t\int_{\Omega}(u^0)^p\mathrm{d}x,$$

where $\tilde{C}>0$ is a large constant depending on ε and δ.

Taking $\varepsilon > 0$ small enough such that $\alpha + 2\varepsilon < \lambda_1(\Omega)$, it follows that

$$u_{\Delta_t},\, \tilde{u}_{\Delta_t} \text{ is bounded on } L^\infty(0,T;L^2(\Omega)), \tag{2.9}$$

$$u_{\Delta_t},\, \tilde{u}_{\Delta_t} \text{ is bounded on } L^p(0,T;W_0^{1,p}(\Omega)). \tag{2.10}$$

We now derive the second energy estimates. Multiplying (2.1) by $(u^n - u^{n-1})$, summing from $n=1$ to N and integrating over Ω, we get

$$\begin{aligned}\Delta_t \sum_{n=1}^N \int_\Omega \Big(\frac{u^n - u^{n-1}}{\Delta_t}\Big)^2 \mathrm{d}x + \sum_{n=1}^N \int_\Omega |\nabla u^n|^{p-2}\nabla u^n \cdot \nabla(u^n - u^{n-1})\mathrm{d}x \\ - \sum_{n=1}^N \int_\Omega \frac{u^n - u^{n-1}}{(u^n)^\delta}\mathrm{d}x \\ \leqslant \sum_{n=1}^N \int_\Omega f(x,u^{n-1})(u^n - u^{n-1})\mathrm{d}x.\end{aligned} \tag{2.11}$$

Therefore, (2.11) together with (2.9) and (0.1) yield

$$\begin{aligned}\frac{\Delta_t}{2}\sum_{n=1}^N \int_\Omega \Big(\frac{u^n - u^{n-1}}{\Delta_t}\Big)^2 \mathrm{d}x \\ + \sum_{n=1}^N \int_\Omega |\nabla u^n|^{p-2}\nabla u^n \cdot \nabla(u^n - u^{n-1})\mathrm{d}x \\ - \sum_{n=1}^N \int_\Omega \frac{u^n - u^{n-1}}{(u^n)^\delta}\mathrm{d}x \leqslant \frac{\Delta_t}{2}\sum_{n=1}^N \int_\Omega f(x,u^{n-1})^2 \mathrm{d}x \leqslant C.\end{aligned} \tag{2.12}$$

From the convexity of the expressions $\int_\Omega |\nabla u|^p \mathrm{d}x$ and $-\frac{1}{1-\delta}\int_\Omega u^{1-\delta}\mathrm{d}x$ we get the following inequalities:

$$\frac{1}{2}\Big[\int_\Omega |\nabla u^n|^p \mathrm{d}x - \int_\Omega |\nabla u^{n-1}|^p \mathrm{d}x\Big] \leqslant \int_\Omega |\nabla u^n|^{p-2}\nabla u^n \nabla(u^n - u^{n-1})\mathrm{d}x$$

and

$$\frac{1}{1-\delta}\Big[\int_\Omega (u^{n-1})^{1-\delta}\mathrm{d}x - \int_\Omega (u^n)^{1-\delta}\mathrm{d}x\Big] \leqslant -\int_\Omega \frac{u^n - u^{n-1}}{(u^n)^\delta}\mathrm{d}x,$$

which imply together with (2.12) and (2.10) that

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} \text{ is bounded on } L^2(Q_T) \text{ uniformly in } \Delta_t. \tag{2.13}$$

From (2.13),

$$\max_{[0,T]} \|\tilde{u}_{\Delta_t}(t) - u_{\Delta_t}(t)\|_{L^2(\Omega)} \leqslant \max_{1\leqslant n\leqslant N} \|u^n - u^{n-1}\|_{L^2(\Omega)} \leqslant C(\Delta_t)^{\frac{1}{2}} \tag{2.14}$$

and

$$u_{\Delta_t}, \tilde{u}_{\Delta_t} \text{ is bounded on } L^\infty(0,T;W_0^{1,p}(\Omega)) \text{ uniformly in } \Delta_t. \tag{2.15}$$

We now use the compactness result of Aubin-Simon (see [23]):

THEOREM 2.1. *Consider* $p\in]1,+\infty[$, $q\in[1,+\infty]$ *and* V,E *and* F *three Banach spaces such that* $V\hookrightarrow\hookrightarrow E\hookrightarrow F$. *Then, if* A *is a bounded subset of* $W^{1,p}(0,T;F)$ *and of* $L^q(0,T;V)$, A *is relatively compact in* $C([0,T],F)$ *and in* $L^p(0,T;E)$.

From Theorem 2.1, (2.13), (2.14) and (2.15), it follows that there exists $u\in C([0,T],L^q(\Omega))$ such that up to a subsequence

u_{Δ_t} and $\tilde{u}_{\Delta_t}$ converge to u in $L^\infty(0,T;L^q(\Omega))$ for any $1\leqslant q<Np/(N-p)$.

Notice that from $p>2N/(N+2)$, we get $2<Np/(N-p)$ and then

$$u_{\Delta_t} \text{ and } \tilde{u}_{\Delta_t} \text{ converge to } u \text{ in } L^\infty(0,T;L^2(\Omega)). \tag{2.16}$$

From (2.13) and (2.15), it follows that

$$u_{\Delta_t},\tilde{u}_{\Delta_t} \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;W_0^{1,p}(\Omega)) \quad\text{and}\quad \frac{\partial\tilde{u}_{\Delta_t}}{\partial t}\rightharpoonup\frac{\partial u}{\partial t} \text{ in } L^2(Q_T)$$

and from (2.14), (0.1) and (2.16), $f(x,u_{\Delta_t}(t-\Delta_t))\to f(x,u)$ in $L^2(Q_T)$.

We now show that u is indeed a solution in the weak sense given in the definition 1.2. From (2.7), multiplying (2.5) by $(u_{\Delta_t}-u)$ and using (2.16), we get by straightforward calculations:

$$\begin{aligned}\int_0^T\int_\Omega\left[\frac{\partial\tilde{u}_{\Delta_t}}{\partial t}-\frac{\partial u}{\partial t}\right](\tilde{u}_{\Delta_t}-u)\mathrm{d}x\mathrm{d}t-\int_0^T\left\langle\Delta_p u_{\Delta_t},u_{\Delta_t}-u\right\rangle\mathrm{d}t&\\ -\int_0^T\int_\Omega u_{\Delta_t}^{-\delta}(u_{\Delta_t}-u)\mathrm{d}x\mathrm{d}t&\\ =\int_0^T\int_\Omega f(x,u_{\Delta_t}(\cdot-\Delta_t))(u_{\Delta_t}-u)\mathrm{d}x\mathrm{d}t+o_{\Delta_t}(1)&\end{aligned}$$

where $o_{\Delta_t}(1)\to 0$ as $\Delta_t\to 0^+$. From the convexity of the term $-\int_\Omega u^{1-\delta}\mathrm{d}x$ and since $u_{\Delta_t}\rightharpoonup u$ in $L^p(0,T;W_0^{1,p}(\Omega))$, we get that

$$\begin{aligned}\int_\Omega|\tilde{u}_{\Delta_t}(T)-u(T)|^2\mathrm{d}x-\int_0^T\left\langle\Delta_p u_{\Delta_t}-\Delta_p u,u_{\Delta_t}-u\right\rangle\mathrm{d}t&\\ -\frac{1}{1-\delta}\int_0^T\int_\Omega\left(u_{\Delta_t}^{1-\delta}-u^{1-\delta}\right)\mathrm{d}x\mathrm{d}t&\\ \leqslant\int_0^T\int_\Omega f(x,u_{\Delta_t}(\cdot-\Delta_t))(u_{\Delta_t}-u)\mathrm{d}x\mathrm{d}t+o_{\Delta_t}(1),&\end{aligned}$$

and from (2.16) we have

$$\int_0^T\int_\Omega f(x,u_{\Delta_t}(\cdot-\Delta_t))(u_{\Delta_t}-u)\mathrm{d}x\mathrm{d}t=o_{\Delta_t}(1)$$

and since $\delta<1$

$$\int_0^T\int_\Omega\left|u_{\Delta_t}^{1-\delta}-u^{1-\delta}\right|\mathrm{d}x\mathrm{d}t=o_{\Delta_t}(1).$$

Then,

$$\frac{1}{2}\int_\Omega |\tilde{u}_{\Delta_t}-u|^2(T)\mathrm{d}x-\int_0^T < \Delta_p u_{\Delta_t}-\Delta_p u, u_{\Delta_t}-u > \mathrm{d}t = o_{\Delta_t}(1).$$

Thus,

$$u_{\Delta_t}\to u \ \text{ in } L^p(0,T;W_0^{1,p}(\Omega)) \quad \text{as } \Delta_t\to 0^+$$

and consequently

$$\Delta_p u_{\Delta_t}\to \Delta_p u \ \text{ in } L^{p'}(0,T;W^{-1,p'}(\Omega)).$$

Notice that $u\geqslant \eta\phi_{1,K}$ in K which implies by Lebesgue theorem that

$$\int_0^T\int_\Omega \frac{1}{(u_{\Delta_t})^\delta} w\mathrm{d}x\mathrm{d}t \to \int_0^T\int_\Omega \frac{1}{u^\delta} w\mathrm{d}x\mathrm{d}t \quad \text{as } \Delta_t\to 0^+$$

for any $w\in C_c^\infty([0,T]\times\Omega)$. Then, passing to the limit as $\Delta_t\to 0^+$ in (2.6), we get from above compactness properties of $\{u_{\Delta_t}\}_{\Delta_t}$ and $\{\tilde{u}_{\Delta_t}\}_{\Delta_t}$ that

$$\int_0^T\int_\Omega \frac{\partial u}{\partial t} w\mathrm{d}x\mathrm{d}t+\int_0^T\int_\Omega \nabla u|^{p-2}\nabla u\nabla w\mathrm{d}x\mathrm{d}t-\int_0^T\int_\Omega \frac{1}{u^\delta} w\mathrm{d}x\mathrm{d}t$$
$$=\int_0^T\int_\Omega f(x,u)w\mathrm{d}x\mathrm{d}t.$$

for any $w\in C_c^\infty([0,T]\times\Omega)$. This completes the proof. □

3. Stabilization in $W_0^{1,p}(\Omega)$

In this section we prove mainly Theorem 1.7. We start with a lemma which gives additional a priori estimates on u_{Δ_t} and $\tilde{u}_{\Delta_t}$ defined in Section 2.

LEMMA 3.1. *Let assumptions in Theorem* 1.4 *be satisfied. Let u be the weak and global solution to* (P_t) *prescribed at* $t=0$ *by* u_0, *given by Theorem* 1.4. *Then:*

1. *there exists* $\underline{u}$ *and* $\overline{u}$ *belonging to* $\mathscr{C}$, *independent of* Δ_t *such that for all* $t\geqslant 0$, $\underline{u}\leqslant u_{\Delta_t}(t),\tilde{u}_{\Delta_t}(t)\leqslant \overline{u}$;
2. $1/u_{\Delta_t}^\delta$ *and* $1/\tilde{u}_{\Delta_t}^\delta$ *are bounded in* $L^\infty(0,\infty;W^{-1,p'}(\Omega))$;
3. $\frac{\partial \tilde{u}_{\Delta_t}}{\partial t}$ *is bounded in* $L^2(0,\infty;L^2(\Omega))$ *independently of* Δ_t;
4. u_{Δ_t}, $\tilde{u}_{\Delta_t}$ *are bounded in* $L^\infty(0,\infty;W_0^{1,p}(\Omega))$.

Proof. We omit the proof of assertion (1) which is contained in the proof of Theorem 1.4 in [3] and assertion (2) follows from assertion (1) and the definiton of $\mathscr{C}$. We need to prove assertions (3) and (4) to complete the proof of lemma 3.1.

Multiplying (2.1) by (u^n-u^{n-1}) and summing from 1 to N' and integrating over Ω, we have

$$\frac{1}{\Delta_t}\sum_{n=1}^{N'}||u^n-u^{n-1}||^2_{L^2(\Omega)}-\sum_{n=1}^{N'}\left\langle \Delta_p u^n, u^n-u^{n-1}\right\rangle$$
$$-\sum_{n=1}^{N'}\int_\Omega \frac{1}{(u^n)^\delta}(u^n-u^{n-1})\mathrm{d}x$$
$$=\Delta_t\sum_{n=1}^{N'}\int_\Omega f(x,u^{n-1})\Big(\frac{u^n-u^{n-1}}{\Delta_t}\Big)\mathrm{d}x.$$

Set $F(x,s)\stackrel{\text{def}}{=}\int_0^s f(x,\tau)\mathrm{d}\tau$. Since f is locally lipchitz with respect to the second variable uniformly in $x\in\Omega$, there exists $R>0$ such that $t\to F(x,t)+\frac{Rt^2}{2}$ is convex in $[0,\|\overline{u}\|_{L^\infty(\Omega)}]$ uniformly in $x\in\Omega$. Then,

$$\sum_{n=1}^{N'}\int_\Omega f(x,u^{n-1})(u^n-u^{n-1})\mathrm{d}x$$
$$=\sum_{n=1}^{N'}\int_\Omega [f(x,u^{n-1})+Ru^{n-1}](u^n-u^{n-1})\mathrm{d}x-\sum_{n=1}^{N'}\int_\Omega Ru^{n-1}(u^n-u^{n-1})\mathrm{d}x$$
$$\leqslant\sum_{n=1}^{N'}\int_\Omega\left[F(x,u^n)-F(x,u^{n-1})+\frac{R}{2}(|u^n|^2-|u^{n-1}|^2)\right]\mathrm{d}x$$
$$-\sum_{n=1}^{N'}\int_\Omega Ru^{n-1}(u^n-u^{n-1})\mathrm{d}x.$$

Using the fact that $a(a-b)=\frac{1}{2}(|a|^2-|b|^2)+\frac{1}{2}|a-b|^2$ for all $a,b\in\mathbb{R}$ we have

$$\sum_{n=1}^{N'}\int_\Omega Ru^{n-1}(u^{n-1}-u^n)=\frac{R}{2}\int_\Omega[|u^0|^2-|u^{N'}|^2]\mathrm{d}x+\frac{R}{2}\sum_{n=1}^{N'}\int_\Omega|u^n-u^{n-1}|^2\mathrm{d}x.$$

From the convexity of the terms $\int_\Omega|\nabla u|^p\mathrm{d}x$ and $-\frac{1}{1-\delta}\int_\Omega u^{1-\delta}\mathrm{d}x$ we derive the following estimates:

$$\frac{1}{p}\left[\int_\Omega|\nabla u^n|^p\mathrm{d}x-\int_\Omega|\nabla u^{n-1}|^p\mathrm{d}x\right]\leqslant\int_\Omega|\nabla u^n|^{p-2}\nabla u^n\nabla(u^n-u^{n-1})\mathrm{d}x,$$
$$\frac{1}{1-\delta}\left[\int_\Omega(u^{n-1})^{1-\delta}\mathrm{d}x-\int_\Omega(u^n)^{1-\delta}\mathrm{d}x\right]\leqslant-\int_\Omega\frac{u^n-u^{n-1}}{(u^n)^\delta}\mathrm{d}x.$$

Gathering the above inequalities, we deduce that

$$\frac{1}{\Delta_t}\sum_{n=1}^{N'}||u^n-u^{n-1}||^2_{L^2(\Omega)}+\int_\Omega\Big[\frac{|\nabla u^{N'}|^p}{p}-\frac{|\nabla u^0|^p}{p}\Big]\mathrm{d}x$$
$$\leqslant\frac{1}{1-\delta}\int_\Omega[|u^{N'}|^{1-\delta}-|u^0|^{1-\delta}]\mathrm{d}x+\int_\Omega\big[F(x,u^{N'})-F(x,u^0)\big]\mathrm{d}x$$
$$-\sum_{n=1}^{N'}\frac{R}{2}\int_\Omega|u^n-u^{n-1}|^2\mathrm{d}x.$$

From the definition of $\mathscr{C}$ and the fact that $\delta < 2 + \frac{1}{p-1}$, notice that

$$\int_\Omega \underline{u}^{1-\delta}\mathrm{d}x < \infty \quad \text{and} \quad \int_\Omega \overline{u}^{1-\delta}\mathrm{d}x < \infty.$$

Consequently, rearranging the terms we have,

$$\begin{aligned}
\frac{1}{\Delta_t}\sum_{n=1}^{N'} ||u^n - u^{n-1}||^2_{L^2(\Omega)} &+ \int_\Omega \Big[\frac{|\nabla u^{N'}|^p}{p} - \frac{|\nabla u^0|^p}{p}\Big]\mathrm{d}x \\
&\leqslant \frac{1}{1-\delta}\int_\Omega {u^{N'}}^{1-\delta}\mathrm{d}x + \int_\Omega \big[F(x,u^{N'}) - F(x,u^0)\big]\mathrm{d}x \\
&\leqslant \int_\Omega \big[F(x,\overline{u}) + R\overline{u}^2\big]\mathrm{d}x + \frac{1}{|1-\delta|}\max\Big(\int_\Omega \overline{u}^{1-\delta}\mathrm{d}x, \int_\Omega \underline{u}^{1-\delta}\mathrm{d}x\Big) \\
&\leqslant C,
\end{aligned}$$

where $C = C(\overline{u},\underline{u},\delta) > 0$ is independent of N' and Δ_t. Thus, we have by the above estimate that $\tilde{u}_{\Delta_t}$ and $\frac{\partial \tilde{u}_{\Delta_t}}{\partial t}$ are bounded respectively in $L^\infty(0,T;W_0^{1,p}(\Omega))$ and in $L^2(Q_T)$ independently of Δ_t and T. Therefore, $\tilde{u}_{\Delta_t}$ and $\frac{\partial \tilde{u}_{\Delta_t}}{\partial t}$ are bounded respectively in $L^\infty(0,\infty;W_0^{1,p}(\Omega))$ and in $L^2(0,\infty;L^2(\Omega))$ independently of Δ_t. □

Now we give the proof of proposition 1.11.

Proof. The fact that

$$\frac{\partial u}{\partial t} \quad \text{belongs to } L^2(0,\infty;L^2(\Omega))$$

is a consequence of assertion (3) in Lemma 3.1. We now prove that

$$\frac{\partial u}{\partial t} \in L^\infty(t_0,\infty;L^2(\Omega)) \quad \text{for any } t_0 > 0.$$

Set

$$\xi_n \overset{\mathrm{def}}{=} \frac{u^n - u^{n-1}}{\Delta_t} \quad \text{for } n \geqslant 1.$$

For $n \geqslant 2$, we subtract equation (2.1) with n from the equation (2.1) substituting n by $n-1$. Multiplying by ξ_n the corresponding equation and integrating on Ω and sum from $N' \geqslant 2$ to $N'' > N'$ we get:

$$\begin{aligned}
\sum_{n=N'}^{N''}\int_\Omega (\xi_n - \xi_{n-1})\xi_n \mathrm{d}x - \sum_{n=N'}^{N''} &< \Delta_p u^n - \Delta_p u^{n-1}, \xi_n > \\
&- \sum_{n=N'}^{N''}\int_\Omega ((\frac{1}{u^n})^\delta - (\frac{1}{u^{n-1}})^\delta)\xi_n \mathrm{d}x \\
&\qquad = \sum_{n=N'}^{N''}\int_\Omega \big[f(x,u^{n-1}) - f(x,u^{n-2})\big]\xi_n \mathrm{d}x.
\end{aligned}$$

Now, by the monotonicity of the operator A, it follows that

$$\sum_{n=N'}^{N''}\int_\Omega(\xi_n-\xi_{n-1})\xi_n\mathrm{d}x \leqslant \Delta_t\sum_{n=N'}^{N''}\int_\Omega\Big[\frac{f(x,u^{n-1})-f(x,u^{n-2})}{\Delta_t}\Big]\xi_n\mathrm{d}x.$$

Using the lipschitz property of f in $[0,|\overline{u}|_\infty]$ and assertion (3) in lemma 3.1, we have that for a constant $C>0$ large enough,

$$\frac{1}{2}\sum_{n=N'}^{N''}\int_\Omega(|\xi_n|^2-|\xi_{n-1}|^2)\mathrm{d}x+\sum_{n=N'}^{N''}\int_\Omega\frac{|\xi^n-\xi^{n-1}|^2}{2}\mathrm{d}x$$

$$\leqslant \Delta_t\sum_{n=N'}^{N''}\omega\int_\Omega[|\xi_{n-1}|^2+|\xi_n|^2]\mathrm{d}x \leqslant C$$

(ω being the lipchitz constant of f in $[0,|\overline{u}|_\infty]$). So finally we have,

$$\frac{1}{2}\int_\Omega[|\xi_{N''}|^2-|\xi_{N'-1}|^2]\mathrm{d}x\leqslant C.$$

We now fix $t_0>0$ and denote by $E(t)$ the integer part of the real number t. We choose $N'\leqslant E(\frac{t_0}{\Delta_t})+1$ such that

$$\begin{aligned}\int_\Omega|\xi_{N'-1}|^2 &= \int_\Omega\Big|\frac{u^{N'-1}-u^{N'-2}}{\Delta_t}\Big|^2\\ &\overset{\text{def}}{=}\min_{n\in[1,\dots,E(\frac{t_0}{\Delta_t})]}\int_\Omega\Big|\frac{u^n-u^{n-1}}{\Delta_t}\Big|^2\\ &\leqslant\frac{\Delta_t}{t_0-\Delta_t}\sum_{1}^{E(\frac{t_0}{\Delta_t})}\int_\Omega\Big|\frac{u^n-u^{n-1}}{\Delta_t}\Big|^2\leqslant C\end{aligned}$$

for $\Delta_t<<t_0$ and by assertion (3) in Lemma 3.1. So for $N''>\frac{t_0}{\Delta_t}$ and $\Delta_t>0$ small enough we get,

$$\int_\Omega\Big|\frac{u^{N''}-u^{N''-1}}{\Delta_t}\Big|^2\leqslant\tilde{C}$$

where $\tilde{C}$ is independent of Δ_t. Then we have that

$$\frac{\partial\tilde{u}_{\Delta_t}}{\partial t}\ \text{ is bounded in }\ L^\infty(t_0,\infty;L^2(\Omega))\ \text{ for } t_0>0.\square$$

From the above estimates, we immediately get the following proposition.

PROPOSITION 3.2. *Let conditions in theorem* 1.4 *be satisfied. Then, for any* $t_0>0$*, up to a subsequence, we have:*

$$u_{\Delta_t},\tilde{u}_{\Delta_t}\overset{*}{\rightharpoonup}u\quad in\ L^\infty(0,\infty;W_0^{1,p}(\Omega))$$

and

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} \overset{*}{\rightharpoonup} \frac{\partial u}{\partial t} \quad \text{in } L^\infty(t_0,\infty;L^2(\Omega)).$$

Next, we have to show the compactness properties of $\{u(t)\}_{t\geqslant t_0}$ for $t_0>0$. Precisely, we prove Proposition 1.12. For that, we recall the following result of J. Simon given in [22] about the problem (3.1) below.

THEOREM 3.3. ([22]) *Let u be the unique solution to*

$$\begin{cases} -\nabla(d|\nabla v|^{p-2}\nabla v) = h & \text{in } \Omega, \\ v = g & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with C^3 regularity, $0<d\in W^{1,\infty}(\Omega)$, $h\in W^{-1,p'}(\Omega)$ and $g\in W^{1-\frac{1}{p},p}(\partial\Omega)$. Then, there exist $M>0$ and $C>0$ independent of h such that:

(i) *if $p>2$, $h\in L^{p'}(\Omega)$ and $g\in W^{2-\frac{1}{p},p}(\partial\Omega)$, then*

$$v\in B_\infty^{1+\frac{1}{(p-1)^2},p}(\Omega) \quad \text{and} \quad \|v\|_{B_\infty^{1+\frac{1}{(p-1)^2},p}(\Omega)} \leqslant M\|h\|_{L^{p'}(\Omega)}+C;$$

(ii) *if $1<p<2$, $h\in B_\infty^{p-2,p'}(\Omega)$ and $g\in B_\infty^{p-\frac{1}{p},p}(\partial\Omega)$, then*

$$v\in B_\infty^{1+(p-1)^2,p}(\Omega) \quad \text{and} \quad \|v\|_{B_\infty^{1+(p-1)^2,p}(\Omega)} \leqslant M\|h\|_{B_\infty^{p-2,p'}(\Omega)}+C.$$

We now prove proposition 1.12.

Proof. Let $t_0>0$. Let us consider first the case $p>2$. Since

$$\frac{\partial u}{\partial t}\in L^\infty(t_0,\infty;L^2(\Omega)) \text{ and } \quad p>2$$

we have

$$\frac{\partial u}{\partial t}\in L^\infty(t_0,\infty;L^{p'}(\Omega))$$

and since

$$\frac{1}{u^\delta}\in L^\infty(0,\infty;L^{p'}(\Omega)) \text{ for } \delta\in\left(0,1-\frac{1}{p}\right)$$

(which follows from $\delta<(p-1)/p$ and the fact that u is uniformly in $\mathscr{C}$) together with Theorem 3.3 we have for $t\geqslant t_0$:

$$\|u(.,t)\|_{B_\infty^{1+\frac{1}{(p-1)^2},p}(\Omega)} \leqslant M\left\|f(.,u)-\frac{\partial u}{\partial t}(.,t)+\frac{1}{u^\delta}\right\|_{L^{p'}(\Omega)}+C<\infty,$$

where M and C are constants independent of t given in Theorem 3.3. This concludes the result for $p>2$. Let us consider finally the case $1<p<2$. From the fact that $N\leqslant 3$

and $N/2 \leqslant p < 2$, we get by Sobolev imbedding and basic Besov spaces properties that $L^2(\Omega)$ is continuously imbedded in $W^{p-2,p'}(\Omega)$ and that $W^{p-2,p'}(\Omega)$ is continuously imbedded in $B_{\infty}^{p-2,p'}(\Omega)$. Since

$$\frac{\partial u}{\partial t} \in L^{\infty}(t_0,\infty; L^2(\Omega)) \quad \text{and} \quad \frac{1}{u^{\delta}} \in L^{\infty}(t_0,\infty; L^2(\Omega))$$

(which follows from $\delta < \frac{1}{2}$ and the fact that u is uniformly in $\mathscr{C}$) we get by applying assertion (ii) of Theorem 3.3, for $t \geqslant t_0$:

$$\|u(.,t)\|_{B_{\infty}^{1+(p-1)2,p}(\Omega)} \leqslant M \left\| f(.,u) - \frac{\partial u}{\partial t}(.,t) + \frac{1}{u^{\delta}} \right\|_{L^2(\Omega)} + C < \infty. \square$$

Finally, we give the proof of Theorem 1.10.

Proof. Using Proposition 1.12 and the compact imbeddings of

$$B_{\infty}^{1+(p-1)^2,p}(\Omega) \text{ and } B_{\infty}^{1+\frac{1}{(p-1)^2},p}(\Omega) \text{ in } W^{1,p}(\Omega),$$

we get that trajectories are relatively compact in $W^{1,p}(\Omega)$ for large t. Combining this compactness property and Theorem 1.7, we obtain (1.5). $\square$

REFERENCES

[1] C. ARANDA AND T. GODOY, *Existence and multiplicity of positive solutions for a singular problem associated to the p-Laplacian operator*, Electron. J. Differential Equations, **132** (2004), 1–15.

[2] R. ARIS, *Mathematical modelling techniques*, Vol. **24** of Research Notes in Mathematics, Pitman (Advanced Publishing Program), Boston, Mass., 1979.

[3] M. BADRA, K. BAL AND J. GIACOMONI, *A singular parabolic equation: existence, stabilization*, to appear.

[4] H.T. BANKS, *Modeling and control in the biomedical sciences*, Vol. **6** Lecture Notes in Biomathematics, Springer-Verlag, Berlin, 1975.

[5] L. BOCCARDO AND L. ORSINA, *Semilinear elliptic equations with singular nonlinearities*, Calc. Var. Partial Differential Equations, **37**, (3-4) (2010), 363–380.

[6] H. BREZIS, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011.

[7] M.G. CRANDALL, P.H. RABINOWITZ AND L. TARTAR, *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations, **2**, (2) (1977), 193–222.

[8] J. DÁVILA AND M. MONTENEGRO, *Existence and asymptotic behavior for a singular parabolic equation*, Trans. Amer. Soc., **357** (5) (2005), 1801–1828.

[9] J.I. DÍAZ, *Nonlinear partial differential equations in free boundaries. Vol. I*, volume **106** of Research Notes in Mathematics, Pitman (Advanced Publishing Program), Boston, M.A., 1985.

[10] J.I. DÍAZ AND J. E. SAÁ *Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires*, C.R. Acad. Sci. Paris Sér. I. Math., **305** (12) (1987), 521–524.

[11] M. GHERGU AND V.D. RADULESCU, *Multi-parameter bifurcation and asymptotics behavior for the singular Lane-Emden-Fowler equation with a convection term*, Proc. Roy. Soc. Edinburgh Sect. A, **135**, (1) (2005), 61–83.

[12] M. GHERGU AND V.D. RADULESCU, *Singular elliptic problems: bifurcation and asymptotic analysis*, volume **37** of Oxford Lecture Series in Mathematics and its Applications. The Clarendon Press Oxford University Press, Oxford, 2008.

[13] J. GIACOMONI, I. SCHINDLER AND P. TAKÁČ, *Sobolev versus Hölder local minimizers and existence of multiple solutions for a singular quasilinear equation*, Ann. Sc. Norm. Super. Pisa Cl Sci. (5), **6**, (1) (2007), 117–158.

[14] J. HERNÁNDEZ, F. MANCEBO, *Singular elliptic and parabolic equations*, Handbook of Differential Equations, **3** (2006), 317–400.

[15] J. HERNÁNDEZ, F. MANCEBO AND J.M. VEGA, *On the linearization of some singular, nonlinear elliptic problems and applications*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **19**, (6) (2002), 777–813.

[16] H.B. KELLER AND D.S. COHEN, *Some positone problems suggested by nonlinear heat generation*, J. Math. Mech., **16** (1967), 1361–1376.

[17] C.D. LUNING AND W.L. PERRY, *Positive solutions of negative exponent generalized Emden-Fowler boundary value problems*, SIAM J. Math. Anal., **12**, (6) (1981), 874–879.

[18] A. NACHMAN AND A. CALLEGARI, *A nonlinear singular boundary value problem in the theory of pseudoplastic fluids*, SIAM J. Appl. Math., **38**, (2) (1980), 275–281.

[19] D. O'REGAN, *Some general existence principles and results for* $(\phi(y'))' = qf(t,y,y'),\quad 0<t<1$, SIAM J. Math. Anal., **24**, (3) (1993), 648–668.

[20] K. PERERA AND E.A.B. SILVA, *On singular* p*-Laplacian problems*, Differential Integral Equations, **20**, (1) (2007), 105–120.

[21] W.L. PERRY, *A monotone iterative technique for solution of* p*-order (* $p<0$ *) reaction-diffusion problems in permeable catalysis*, J. Comput. Chem., **5**, (4) (1984), 353–357.

[22] J. SIMON, *Régularité de la solution d'un problème aux limites nonlinéaires*, Ann. Fac. Sci. Toulouse Math. (5), **3**, (3-4) (1981), 247–274.

[23] J. SIMON, *Compact sets in the space* $L^p(0,T;B)$, Ann. Mat. Pura Appl. (4), **146** (1987), 65–96.

[24] P. TAKÁČ, *Stabilization of positive solutions for analytic gradient-like systems*, Discrete Contin. Dynam. Systems, **6**, (4) (2000), 947–973.

[25] H. TRIEBEL, *Interpolation theory, functions spaces, differential operators*, Johann Ambrosius Barth, Heidelberg, second edition, 1995.

[26] M. WINKLER, *Nonuniqueness in the quenching problem*, Math. Ann., **339** (3) (2007), 559–597.

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Mehdi Badra
LMAP (UMR 5142), Bat. IPRA
Université de Pau et des Pays de l'Adour
Avenue de l'Université, 64013 cedex Pau
France
e-mail: mehdi.badra@univ-pau.fr

Kaushik Bal
LMAP (UMR 5142), Bat. IPRA
Université de Pau et des Pays de l'Adour
Avenue de l'Université, 64013 cedex Pau
France
e-mail: kaushik.bal@univ-pau.fr

Jacques Giacomoni
LMAP (UMR 5142), Bat. IPRA
Université de Pau et des Pays de l'Adour
Avenue de l'Université, 64013 cedex Pau
France
e-mail: jgiacomo@univ-pau.fr

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www.ele-math.com
dea@ele-math.com