
A singular parabolic equation: Existence, stabilization

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ABSTRACT

We investigate the following quasilinear parabolic and singular equation,

$$(P_t) \begin{cases} u_t - \Delta_p u = \frac{1}{u^\delta} + f(x, u) & \text{in } (0, T) \times \Omega, \\ u = 0 & \text{on } (0, T) \times \partial\Omega, \quad u > 0 & \text{in } (0, T) \times \Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega, \end{cases}$$

where Ω is an open bounded domain with smooth boundary in \mathbb{R}^N , $1 < p < \infty$, $0 < \delta$ and $T > 0$. We assume that $(x, s) \in \Omega \times \mathbb{R}^+ \rightarrow f(x, s)$ is a bounded below Caratheodory function, locally Lipschitz with respect to s uniformly in $x \in \Omega$ and asymptotically sub-homogeneous, i.e.

$$0 \leq \lim_{t \rightarrow +\infty} \frac{f(x, t)}{t^{p-1}} = \alpha_f < \lambda_1(\Omega) \quad (0.1)$$

(where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta_p$ in Ω with homogeneous Dirichlet boundary conditions) and $u_0 \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$, satisfying a cone condition defined below. Then, for any $\delta \in (0, 2 + \frac{1}{p-1})$, we prove the existence and the uniqueness of a weak solution $u \in \mathbf{V}(Q_T)$ to (P_t) . Furthermore, $u \in C([0, T], W_0^{1,p}(\Omega))$ and the restriction $\delta < 2 + \frac{1}{p-1}$ is sharp. The proof relies on a semi-discretization in time with implicit Euler method and on the study of the stationary problem. The key points in the proof is to show that u belongs to the cone \mathcal{C} defined below and by the weak comparison principle that $\frac{1}{u^\delta} \in L^\infty(0, T; W^{-1,p'}(\Omega))$ and $u^{1-\delta} \in L^\infty(0, T; L^1(\Omega))$. When $t \rightarrow \frac{f(x,t)}{t^{p-1}}$ is nonincreasing for a.e. $x \in \Omega$, we show that $u(t) \rightarrow u_\infty$ in $L^\infty(\Omega)$ as $t \rightarrow \infty$, where u_∞ is

the unique solution to the stationary problem. This stabilization property is proved by using the accretivity of a suitable operator in $L^\infty(\Omega)$.

Finally, in the last section we analyze the case $p = 2$. Using the interpolation spaces theory and the semigroup theory, we prove the existence and the uniqueness of weak solutions to (P_t) for any $\delta > 0$ in $C([0, T], L^2(\Omega)) \cap L^\infty(Q_T)$ and under suitable assumptions on the initial data we give additional regularity results. Finally, we describe their asymptotic behaviour in $L^\infty(\Omega) \cap H_0^1(\Omega)$ when $\delta < 3$.

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0. Introduction

In the present paper we investigate the following quasilinear and singular parabolic problem:

$$(P_t) \quad \begin{cases} u_t - \Delta_p u = \frac{1}{u^\delta} + f(x, u) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \quad u > 0 & \text{in } Q_T, \\ u(0, x) = u_0(x) & \text{in } \Omega, \end{cases}$$

where Ω is an open bounded domain with smooth boundary in \mathbb{R}^N (with $N \geq 2$), $1 < p < \infty$, $0 < \delta$, $T > 0$, $Q_T = (0, T) \times \Omega$ and $\Sigma_T = (0, T) \times \partial\Omega$. We assume that f is a bounded below Caratheodory function, locally Lipschitz with respect to the second variable uniformly in $x \in \Omega$ and satisfying (0.1) and $u_0 \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$. Such a problem with $p = 2$ arises in the study of non-Newtonian fluids (in particular *pseudoplastic* fluids), boundary-layer phenomena for viscous fluids (see [14,31,30]), in the Langmuir–Hinshelwood model of chemical heterogeneous catalyst kinetics (see [4,35]), in enzymatic kinetics models (see [5]), as well as in the theory of heat conduction in electrically conducting materials (see [26]) and in the study of guided modes of an electromagnetic field in nonlinear medium (see [19]). Problem (P_t) with $p \neq 2$ arises specifically in the study of turbulent flow of a gas in porous media (see [32]). We refer to the survey Hernández, Mancebo and Vega [25], the book Ghergu and Rădulescu [20] and the bibliography therein for more details about the corresponding models. One of our main goals is to prove the existence and the uniqueness of the weak solution to (P_t) and to discuss its global behaviour. In particular, it is important in the applications for the above models to prove stabilization phenomena (i.e. convergence to a steady state as $t \rightarrow \infty$). We define the notion of weak solution for the following more general problem

$$(S_t) \quad \begin{cases} u_t - \Delta_p u = \frac{1}{u^\delta} + h(x, t) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \quad u > 0 & \text{in } Q_T, \\ u(0, x) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $0 < T$, $h \in L^\infty(Q_T)$, $0 < \delta < 2 + \frac{1}{p-1}$, $u_0 \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$, as follows:

Definition 0.1.

$$\mathbf{V}(Q_T) \stackrel{\text{def}}{=} \{u: u \in L^\infty(Q_T), u_t \in L^2(Q_T), u \in L^\infty(0, T; W_0^{1,p}(\Omega))\}$$

and

Definition 0.2. A weak solution to (S_t) is a function $u \in \mathbf{V}(Q_T)$ satisfying

1. for any compact $K \subset Q_T$, $\text{ess\,inf}_K u > 0$,
2. for every test function $\phi \in \mathbf{V}(Q_T)$,

$$\int_{Q_T} \left(\phi \frac{\partial u}{\partial t} + |\nabla u|^{p-2} \nabla u \nabla \phi - \phi \left(\frac{1}{u^\delta} + h(t, x) \right) \right) dx dt = 0,$$

3. $u(0, x) = u_0(x)$ a.e. in Ω .

Remark 0.3. If $\frac{1}{u^\delta} \in L^\infty(0, T; W^{-1,p'}(\Omega))$, then the second point of Definition 0.2 makes sense.

Remark 0.4. Since every $u \in \mathbf{V}(Q_T)$ belongs to $C(0, T; L^2(\Omega))$, the third point of the above definition is meaningful.

The approach we use is to study first the existence of solutions to the stationary problem (P) that is for $g \in L^\infty(\Omega)$, $\lambda > 0$

$$(P) \quad \begin{cases} u - \lambda \left(\Delta_p u + \frac{1}{u^\delta} \right) = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

To control the singular term $\frac{1}{u^\delta}$, we need to consider solutions in a conical shell \mathcal{C} defined as the set of functions $v \in L^\infty(\Omega)$ such that there exist $c_1 > 0$ and $c_2 > 0$ satisfying

$$\begin{cases} c_1 d(x) \leq v \leq c_2 d(x) & \text{if } \delta < 1, \\ c_1 d(x) \log^{\frac{1}{p}} \left(\frac{k}{d(x)} \right) \leq v \leq c_2 d(x) \log^{\frac{1}{p}} \left(\frac{k}{d(x)} \right) & \text{if } \delta = 1, \\ c_1 d(x)^{\frac{p}{\delta+p-1}} \leq v \leq c_2 (d(x)^{\frac{p}{\delta+p-1}} + d(x)) & \text{if } \delta > 1, \end{cases}$$

where $d(x) \stackrel{\text{def}}{=} \text{dist}(x, \partial\Omega)$ and $k > 0$ is large enough. Regarding problem (P), we prove the following

Theorem 0.5. Let $g \in L^\infty(\Omega)$ and $0 < \delta < 2 + \frac{1}{p-1}$. Then for any $\lambda > 0$, problem (P) admits a unique solution u_λ in $W_0^{1,p}(\Omega) \cap \mathcal{C} \cap C_0(\overline{\Omega})$.

Concerning the case where $\delta \geq 2 + \frac{1}{p-1}$, we prove the following

Theorem 0.6. Let $g \in L^\infty(\Omega)$ and $\delta \geq 2 + \frac{1}{p-1}$. Then for any $\lambda > 0$, problem (P) admits a solution u_λ in $W_{\text{loc}}^{1,p}(\Omega) \cap \mathcal{C} \cap C_0(\overline{\Omega})$ such that $u_\lambda \notin W_0^{1,p}(\Omega)$.

Remark 0.7. Using a similar approach as in [8], it can be shown in this case that for some $\gamma = \gamma(\delta) > 1$, $u_\lambda^\gamma \in W_0^{1,p}(\Omega)$.

In view of establishing Theorem 0.15 below, we need to prove the following result:

Theorem 0.8. Let $0 < \delta < 2 + \frac{1}{p-1}$ and $f : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}$ be a bounded below Caratheodory function, locally Lipschitz with respect to the second variable uniformly in $x \in \Omega$, satisfying (0.1) and such that $\frac{f(x,s)}{s^{p-1}}$ is a decreasing function in \mathbb{R}^+ for a.e. $x \in \Omega$. Then there exists a unique u_∞ in $W_0^{1,p}(\Omega) \cap C \cap C_0(\overline{\Omega})$ satisfying

$$(Q) \quad \begin{cases} -\Delta_p u_\infty - \frac{1}{u_\infty^\delta} = f(x, u_\infty) & \text{in } \Omega, \\ u_\infty = 0 & \text{on } \partial\Omega. \end{cases}$$

Using a semi-discretization in time with implicit Euler method, Theorem 0.5, energy estimates and the weak comparison principle (see Cuesta and Takáč [10], Fleckinger-Pellé and Takáč [18]), we prove the following

Theorem 0.9. Let $0 < \delta < 2 + \frac{1}{p-1}$, $h \in L^\infty(Q_T)$ and $u_0 \in W_0^{1,p}(\Omega) \cap C$. Then there exists a unique weak solution u to

$$(0.2) \quad \begin{cases} u_t - \Delta_p u = \frac{1}{u^\delta} + h(x, t) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \quad u > 0 & \text{in } Q_T, \\ u(0, x) = u_0(x) & \text{in } \Omega, \end{cases}$$

such that $u(t) \in C$ uniformly for $t \in [0, T]$. Moreover, u belongs to $C([0, T], W_0^{1,p}(\Omega))$ and satisfies for any $t \in [0, T]$:

$$(0.3) \quad \begin{aligned} & \int_0^t \int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 dx ds + \frac{1}{p} \int_\Omega |\nabla u(t)|^p dx - \frac{1}{1-\delta} \int_\Omega u^{1-\delta}(t) dx \\ &= \int_0^t \int_\Omega h \frac{\partial u}{\partial t} dx ds + \frac{1}{p} \int_\Omega |\nabla u_0|^p dx - \frac{1}{1-\delta} \int_\Omega u_0^{1-\delta} dx. \end{aligned}$$

Remark 0.10. Saying that $u(t) \in C$ uniformly for $t \in [0, T]$ means that there exist $\underline{u}, \bar{u} \in C$ such that $\underline{u}(x) \leq u(t, x) \leq \bar{u}(x)$ a.e. $(x, t) \in \Omega \times [0, T]$.

Remark 0.11. By Theorem 0.6, the restriction $\delta < 2 + \frac{1}{p-1}$ is sharp.

Moreover, we have the following regularity result for solutions to (0.2) which is obtained from the theory of nonlinear monotone operators of [6]:

Proposition 0.1. Assume that hypotheses of Theorem 0.9 are satisfied and set

$$C_0(\overline{\Omega}) \stackrel{\text{def}}{=} \{v \in C(\overline{\Omega}) \mid u = 0 \text{ on } \partial\Omega\}$$

and

$$D(A) \stackrel{\text{def}}{=} \left\{ v \in C \cap W_0^{1,p}(\Omega) \mid Av \stackrel{\text{def}}{=} -\Delta_p v - \frac{1}{v^\delta} \in L^\infty(\Omega) \right\}. \tag{0.4}$$

If in addition $u_0 \in \overline{D(A)}^{L^\infty(\Omega)}$, then the solution u to (0.2) belongs to $C([0, T]; C_0(\overline{\Omega}))$ and satisfies:

(i) If v is another weak solution to (S_t) with initial datum $v_0 \in \overline{\mathcal{D}(A)}^{L^\infty(\Omega)}$ and nonhomogeneous term $k \in L^\infty(Q_T)$ then the following estimate holds

$$\|u(t) - v(t)\|_{L^\infty(\Omega)} \leq \|u_0 - v_0\|_{L^\infty(\Omega)} + \int_0^t \|h(s) - k(s)\|_{L^\infty(\Omega)} ds, \quad 0 \leq t \leq T. \quad (0.5)$$

(ii) If $u_0 \in \mathcal{D}(A)$ and $h \in W^{1,1}(0, T; L^\infty(\Omega))$ then $u \in W^{1,\infty}(0, T; L^\infty(\Omega))$ and $\Delta_p u + u^{-\delta} \in L^\infty(Q_T)$, and the following estimate holds

$$\left\| \frac{du(t)}{dt} \right\|_{L^\infty(\Omega)} \leq \|\Delta_p u_0 + u_0^{-\delta} + h(0)\|_{L^\infty(\Omega)} + \int_0^T \left\| \frac{dh(t)}{dt} \right\|_{L^\infty(\Omega)} d\tau. \quad (0.6)$$

Remark 0.12. Assertion (i) of Proposition 0.1 can be obtained from [6, Thm. 4.2, p. 130] and assertion (ii) can be shown by using arguments in the proof of [6, Thm. 4.4, p. 141]. Since $L^\infty(\Omega)$ is not reflexive, we need additional arguments to establish (0.6) (see in particular the end of the proof of Proposition 0.1). For the sake of clarity, we give complete proofs of assertions (i) and (ii) in Section 2.

Concerning problem (P_t) , we have the following

Theorem 0.13. Let $0 < \delta < 2 + \frac{1}{p-1}$. Assume that f is a bounded below Caratheodory function, and that f is locally Lipschitz with respect to the second variable uniformly in $x \in \Omega$ and satisfying (0.1). Let $u_0 \in W_0^{1,p}(\Omega) \cap C$. Then, for any $T > 0$, there exists a unique weak solution, u , to (P_t) such that $u(t) \in C$ uniformly for $t \in [0, T]$, $u \in C([0, T], W_0^{1,p}(\Omega))$ and u satisfies for any $t \in [0, T]$:

$$\begin{aligned} & \int_0^t \int_\Omega \left(\frac{\partial u}{\partial t} \right)^2 dx ds + \frac{1}{p} \int_\Omega |\nabla u(t)|^p dx - \frac{1}{1-\delta} \int_\Omega u^{1-\delta}(t) dx \\ &= \int_\Omega F(x, u(t)) dx + \frac{1}{p} \int_\Omega |\nabla u_0|^p dx - \frac{1}{1-\delta} \int_\Omega u_0^{1-\delta} dx - \int_\Omega F(x, u_0) dx, \end{aligned} \quad (0.7)$$

where $F(x, w) \stackrel{\text{def}}{=} \int_0^w f(x, s) ds$.

A straightforward application of Proposition 0.1 yields the following

Proposition 0.2. Assume that conditions in Theorem 0.13 are satisfied. If in addition $u_0 \in \overline{\mathcal{D}(A)}^{L^\infty(\Omega)}$, then the solution u to (P_t) belongs to $C([0, T]; C_0(\overline{\Omega}))$ and:

(i) There exists $\omega > 0$ such that if v is another weak solution to (P_t) with initial datum $v_0 \in \overline{\mathcal{D}(A)}^{L^\infty(\Omega)}$ then the following estimate holds

$$\|u(t) - v(t)\|_{L^\infty(\Omega)} \leq e^{\omega t} \|u_0 - v_0\|_{L^\infty(\Omega)}, \quad 0 \leq t \leq T.$$

(ii) If $u_0 \in \mathcal{D}(A)$ then $u \in W^{1,\infty}(0, T; L^\infty(\Omega))$ and $\Delta_p u + u^{-\delta} \in L^\infty(Q_T)$, and the following estimate holds

$$\left\| \frac{du(t)}{dt} \right\|_{L^\infty(\Omega)} \leq e^{\omega t} \|\Delta_p u_0 + u_0^{-\delta} + f(x, u_0)\|_{L^\infty(\Omega)}.$$

Remark 0.14. The constant ω given above is equal to the Lipschitz constant of $f(x, \cdot)$ on $[\underline{u}, \bar{u}]$ where \underline{u} and \bar{u} are respectively subsolution and supersolution to (Q) given in (3.2) and (3.3) below.

From Theorems 0.13 and 0.8, we can show the following asymptotic behaviour for solutions to (P_t) :

Theorem 0.15. Let hypothesis in Theorem 0.13 satisfied and assume that $\frac{f(x,s)}{s^{p-1}}$ is decreasing in $(0, \infty)$ for a.e. $x \in \Omega$. Then, the solution to (P_t) is defined in $(0, \infty) \times \Omega$ and satisfies

$$u(t) \rightarrow u_\infty \quad \text{in } L^\infty(\Omega) \text{ as } t \rightarrow \infty, \tag{0.8}$$

where u_∞ is defined in Theorem 0.8.

Concerning the non-degenerate case, i.e. $p = 2$, we can give additional results. In particular, we prove the existence of solutions in the sense of distributions for any $0 < \delta$. Precisely,

Theorem 0.16. Let $0 < \delta$ and $p = 2$. Let f satisfy assumptions in Theorem 0.13 and $u_0 \in C$. Then, for any $T > 0$, there exists a unique solution $u \in C([0, T], L^2(\Omega)) \cap L^\infty(Q_T)$ to (P_t) in the sense of distributions, that is $u \in C$ uniformly in $t \in [0, T]$ and for any $\phi \in \mathcal{D}(Q_T)$, we have

$$-\int_{Q_T} u \frac{\partial \phi}{\partial t} \, dx \, dt - \int_{Q_T} u \Delta \phi \, dx \, dt = \int_{Q_T} \left(\frac{1}{u^\delta} + f(x, u) \right) \phi \, dx \, dt. \tag{0.9}$$

In addition, we have for $0 < \eta$ small enough, the following regularity property:

- (i) if $\delta < \frac{1}{2}$ and $u_0 \in C \cap H^{2-\eta}(\Omega)$, then $u \in C([0, T]; H^{2-\eta}(\Omega))$;
- (ii) if $\frac{1}{2} \leq \delta < 1$ and $u_0 \in C \cap H^{\frac{5}{2}-\delta-\eta}(\Omega)$, then $u \in C([0, T], H^{\frac{5}{2}-\delta-\eta}(\Omega))$;
- (iii) if $1 \leq \delta$ and $u_0 \in C \cap H^{\frac{1}{2}+\frac{2}{\delta+1}-\eta}(\Omega)$, then $u \in C([0, T], H^{\frac{1}{2}+\frac{2}{\delta+1}-\eta}(\Omega))$.

Moreover,

- (iv) if u_1, u_2 are solutions corresponding to initial data $u_{1,0} \in C, u_{2,0} \in C$ respectively, then there exist \underline{u}, \bar{u} in C and a positive constant ω (proportional to the Lipschitz constant of $f(x, \cdot)$ in $[0, \|\bar{u}\|_\infty]$) such that

$$\|u_1(t) - u_2(t)\|_{L^2(\Omega)} \leq e^{(\omega-\lambda_1)t} \|u_{1,0} - u_{2,0}\|_{L^2(\Omega)} \quad \text{and} \quad \underline{u} \leq u^i \leq \bar{u}, \quad i = 1, 2; \tag{0.10}$$

- (v) if $f(x, \cdot)$ is a nonincreasing function then (0.10) is true with $\omega = 0$. Then, the solution to (P_t) , u , defined in $(0, \infty)$ satisfies $u(t) \rightarrow u_\infty$ as $t \rightarrow +\infty$ in $L^2(\Omega)$ where u_∞ is the solution given in Theorem 0.8.

Remark 0.17. In particular, if $\delta < 3$ and $u_0 \in H_0^1(\Omega)$, then we recover $u \in C([0, T]; H_0^1(\Omega))$. Note also that for arbitrary $\delta > 0$ there is $\epsilon > 0$ such that $u \in C([0, T]; H_0^{\frac{1}{2}+\epsilon}(\Omega))$ if $u_0 \in H_0^{\frac{1}{2}+\epsilon}(\Omega)$.

Theorem 0.16 is established using the interpolation theory in Sobolev spaces and the L^p - L^q -maximal regularity results of the linear heat equation. Under the assumptions given in Theorem 0.8, we can derive from Theorem 0.16 some stabilization properties. Precisely, we prove

Theorem 0.18. Let $p = 2, \delta < 3, u_0 \in C \cap H_0^1(\Omega)$. Assume that f satisfies assumptions of Theorem 0.15. Then, the solution to (P_t) , u , defined in $(0, \infty) \times \Omega$ satisfies $u(t) \rightarrow u_\infty$ as $t \rightarrow +\infty$ in $L^\infty(\Omega) \cap H_0^1(\Omega)$ where u_∞ is the solution given in Theorem 0.8.

We now give briefly the state of art concerning parabolic quasilinear singular equations. The corresponding stationary equation was studied intensively in the literature. In particular the case $p = 2$, mostly when $\delta < 1$ and under different assumptions on the asymptotic behaviour of f was considered in detail (see the pioneering work Crandall, Rabinowitz and Tartar [9], the bibliography in Hernández and Mancebo [24]). In our knowledge, the quasilinear case, namely $p \neq 2$, was not considered so far. First, we would like to quote the work Agarwal, Lü and O'Regan [29] where existence of solutions to the one-dimensional problem is obtained via O.D.E. techniques. Next, we mention the work Aranda and Godoy [3] where existence results are obtained via the bifurcation theory for $1 < p \leq 2$ and $f(x, u) = g(u)$ satisfying some growth conditions. In Giacomoni, Schindler and Takáč [21] the existence and multiplicity results when $1 < p < \infty$, $f(x, u) = u^q$ with $p - 1 < q \leq p^* - 1$ and $0 < \delta < 1$ are proved by using variational methods and regularity results in Hölder spaces. In Perera and Silva [34], other kinds of singularities are investigated (for instance $e^{\frac{1}{u}}$ instead of $\frac{1}{u^\delta}$). In Boccardo and Orsina [8], nonexistence results are proved for quasilinear equations involving singular terms in the form $\frac{q(x)}{u^\delta}$ where q belongs to a certain class of bounded Radon measure (for instance a Dirac mass). Concerning the parabolic case, available results mostly concern the case $p = 2$. Namely, we first quote the result in Hernández, Mancebo and Vega [25] where properties of the linearized operator (in $C_0^1(\overline{\Omega})$) and the validity of the strong maximum principle are given, that induce the asymptotic stability of a certain class of stationary solutions in the range $0 < \delta < \frac{1}{2}$. In Takáč [38], a stabilization result in C^1 is proved for a similar class of parabolic singular problems via a clever use of weighted Sobolev spaces. Notice that the common feature of these two works is that solutions belong to $C(0, \infty; [C_0^1(\overline{\Omega})]^+)$, where $[C_0^1(\overline{\Omega})]^+$ denotes the interior of the positive cone of $C_0^1(\overline{\Omega})$ which gives an implicit control of the solution near the boundary $\partial\Omega$. However, in the context of problem (P_t) this approach fails for large δ (that is for $\delta \geq 1$) since weak solutions do not belong to $C^1(\overline{\Omega})$. In the present paper, to deal with the case $\delta \geq 1$, we introduce a new approach by considering the nonlinear operator A instead of the second order diffusion operator $-\Delta_p$ only and taking advantage of the monotonicity of A , by showing the invariance of a conical shell, namely \mathcal{C} , along the flow associated to (P_t) . We highlight that by the weak comparison principle, \mathcal{C} belongs to the closure of the domain of A which is m -accretive in suitable spaces ($L^\infty(\Omega)$, $(\mathcal{D}((-\Delta)^\theta))'$ with $0 < \theta < 1$ for $p = 2$). We stress that these properties (in particular the properties of the resolvent operator associated to A) were not brought out in previous works and are exploited in the present work to prove existence and stabilization of weak solutions for any $0 < \delta < 2 + \frac{1}{p-1}$ (see Theorems 0.13, 0.15) and in the special case $p = 2$ for any $\delta > 0$ (see Theorem 0.16).

We also mention the work Dávila and Montenegro [11] still concerning the case $p = 2$ and with singular absorption term. In this nice work, the authors achieved uniqueness within the class of functions satisfying $u(x, t) \geq c \operatorname{dist}(x, \partial\Omega)^\gamma$ for suitable γ and $c > 0$ and discuss the asymptotic behaviour of solutions. Finally, we would like to quote the nice paper Winkler [43] where the author shows that uniqueness is violated in case of nonhomogeneous boundary Dirichlet condition.

The present paper is organized as follows. The two next sections (Section 1, Section 2) contain the proofs of Theorems 0.5, 0.6, 0.9 and Proposition 0.1. Theorems 0.8, 0.13, 0.15 and Proposition 0.2 are established in Section 3. Finally, the non-degenerate case (i.e. $p = 2$) is dealt in Section 4 where in particular Theorems 0.16 and 0.18 are proved. We stress that the methods used to prove these results are not specific to the special form of equation (P_t) and can be used for equations involving more general class of singular nonlinearities.

1. Proof of Theorems 0.5 and 0.6

We first prove Theorem 0.5.

Proof of Theorem 0.5. First, let us consider the case $\delta < 1$. For $\lambda > 0$, we define the following energy functional:

$$E_\lambda(u) \stackrel{\text{def}}{=} \frac{1}{2} \int_{\Omega} u^2 \, dx + \frac{\lambda}{p} \int_{\Omega} |\nabla u|^p \, dx - \frac{\lambda}{1-\delta} \int_{\Omega} (u^+)^{1-\delta} \, dx - \int_{\Omega} gu^+ \, dx.$$

E_λ is well defined in $X = W_0^{1,p}(\Omega)$ if $p \geq \frac{2N}{N+2}$. If $1 < p < \frac{2N}{N+2}$, E_λ is well defined in $X = W_0^{1,p}(\Omega) \cap L^2(\Omega)$. It is easy to see that E_λ is continuous, coercive in X and strictly convex on the positive cone of X . Thus, since X is reflexive and since $E_\lambda(u^+) \leq E_\lambda(u)$ for any $u \in X$, E_λ admits a unique global minimizer denoted by u_λ and $u_\lambda \geq 0$ a.e. in Ω . We show now that $u_\lambda \in C$. Let ϕ_1 be the normalized positive eigenfunction associated with the principal eigenvalue $\lambda_1(\Omega)$ of $-\Delta_p$ with homogeneous boundary Dirichlet conditions (see Anane [1,2] for further details):

$$-\Delta_p \phi_1 = \lambda_1 |\phi_1|^{p-2} \phi_1 \quad \text{in } \Omega; \quad \phi_1 = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

$\phi_1 \in W_0^{1,p}(\Omega)$ is normalized by $\phi_1 > 0$ in Ω and $\int_\Omega \phi_1^p dx = 1$. Note that the strong maximum and boundary point principles from Vázquez [42, Thm. 5, p. 200] guarantee $\phi_1 > 0$ in Ω and $\frac{\partial \phi_1}{\partial \nu} < 0$ on $\partial\Omega$, respectively. Hence, since $\phi_1 \in C^1(\overline{\Omega})$, there are constants ℓ and L , $0 < \ell < L$, such that $\ell d(x) \leq \phi_1(x) \leq Ld(x)$ for all $x \in \Omega$. Moreover, we observe that for $\epsilon > 0$ small enough (depending on λ, δ and g) we have

$$\begin{cases} \epsilon \phi_1 - \lambda \left(\Delta_p(\epsilon \phi_1) + \frac{1}{(\epsilon \phi_1)^\delta} \right) < g & \text{in } \Omega, \\ \epsilon \phi_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Thus, for $t > 0$, we set $v_\lambda \stackrel{\text{def}}{=} (\epsilon \phi_1 - u_\lambda)^+$ and $\chi(t) \stackrel{\text{def}}{=} E_\lambda(u_\lambda + tv_\lambda)$. From the Hardy Inequality, it follows that χ is differentiable for $t \in (0, 1]$ and

$$\chi'(t) = \langle E'_\lambda(u_\lambda + tv_\lambda), v_\lambda \rangle.$$

The optimality of u_λ guarantees $\chi'(0^+) \geq 0$ and the strict convexity of E_λ ensures that $t \rightarrow \chi'(t)$ is increasing. Therefore, with (1.2) we obtain that

$$0 \leq \chi'(1) = \langle E'_\lambda(\epsilon \phi_1), v_\lambda \rangle < 0$$

if v_λ has non-zero measure support. Then $\epsilon \phi_1 \leq u_\lambda$ and E_λ is Gâteaux-differentiable in u_λ . Consequently, for any $\phi \in X$,

$$\langle E'_\lambda(u_\lambda), \phi \rangle = \left\langle u_\lambda - \lambda \left(\Delta_p u_\lambda + \frac{1}{u_\lambda^\delta} \right) - g, \phi \right\rangle = 0.$$

We observe that if $\delta < 2 + \frac{1}{p-1}$ then

$$u \rightarrow u - \lambda \left(\Delta_p u - \frac{1}{u^\delta} \right) \text{ is monotone from } W_0^{1,p}(\Omega) \cap C \text{ to } W^{-1, \frac{p}{p-1}}(\Omega). \tag{1.3}$$

Then, by the weak comparison principle, we have also that

$$u_\lambda \leq M$$

for any $M > |g|_{L^\infty(\Omega)} + \frac{\lambda}{|g|_{L^\infty(\Omega)}^\delta}$. Then, $u_\lambda \in L^\infty(\Omega)$. Let $U \in C^{1,\alpha}(\overline{\Omega}) \cap C$ (with suitable $0 < \alpha < 1$) be the unique positive solution (see Giacomoni, Schindler and Takáč [21, Thm. B.1] for the existence and regularity of U) to

$$\begin{cases} -\Delta_p u = \frac{1}{u^\delta} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.4)$$

Therefore, observing that for $M' > 0$,

$$\begin{cases} M'U - \lambda \left(\Delta_p(M'U) + \frac{1}{(M'U)^\delta} \right) = M'U + \frac{\lambda(M'^{p-1} - M'^{-\delta})}{U^\delta} & \text{in } \Omega, \\ M'U = 0 & \text{on } \partial\Omega, \end{cases}$$

and by the weak comparison principle, we get that $u_\lambda \leq M'U$ for M' large enough. Together with $\epsilon\phi_1 \leq u_\lambda$, it follows that $u_\lambda \in \mathcal{C}$. Again using Giacomoni, Schindler and Takáč [21, Thm. B.1], we get that $u_\lambda \in C^{1,\alpha}(\overline{\Omega})$ and then $u_\lambda \in C_0(\overline{\Omega})$.

We consider now the case $\delta \geq 1$. We use in this case the weak comparison principle, the existence of suitable subsolutions and supersolutions of the following approximated problem:

$$(P_\epsilon) \quad \begin{cases} u - \lambda \left(\Delta_p u + \frac{1}{(u + \epsilon)^\delta} \right) = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \quad u > 0 & \text{in } \Omega. \end{cases}$$

Using a minimization argument as in the case $\delta < 1$, we get the existence and the uniqueness of the solution to (P_ϵ) , denoted u_ϵ , in $W_0^{1,p}(\Omega)^+ \cap L^\infty(\Omega)$. From the elliptic regularity theory (see Lieberman [27]), we obtain that $u_\epsilon \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. We now construct appropriate subsolutions and supersolutions for (P_ϵ) . For $\delta = 1$, by straightforward computations we have that for $A > 0$ large enough (depending on the diameter of Ω), and for $\eta > 0$ small enough (depending on λ and g but not on ϵ)

$$\underline{u}_\epsilon \stackrel{\text{def}}{=} (\eta\phi_1 + \epsilon') \left[\ln \left(\frac{A}{\eta\phi_1 + \epsilon'} \right) \right]^{\frac{1}{p}} - \epsilon' \left[\ln \left(\frac{A}{\epsilon'} \right) \right]^{\frac{1}{p}}, \quad (1.5)$$

with $\epsilon' > 0$ satisfying $\epsilon = \epsilon' \left[\ln \left(\frac{A}{\epsilon'} \right) \right]^{\frac{1}{p}}$, is a subsolution to (P_ϵ) . Similarly, for $M > 0$ large enough (depending on λ and g but not on ϵ)

$$\bar{u}_\epsilon \stackrel{\text{def}}{=} (M\phi_1 + \epsilon') \left[\ln \left(\frac{A}{M\phi_1 + \epsilon'} \right) \right]^{\frac{1}{p}} - \epsilon' \left[\ln \left(\frac{A}{\epsilon'} \right) \right]^{\frac{1}{p}}, \quad (1.6)$$

is a supersolution to (P_ϵ) satisfying $\bar{u}_\epsilon \geq \underline{u}_\epsilon$. If $\delta > 1$, we consider the following subsolution and supersolution respectively:

$$\underline{u}_\epsilon \stackrel{\text{def}}{=} \eta \left[\left(\phi_1 + \epsilon \frac{p-1+\delta}{p} \right)^{\frac{p}{p-1+\delta}} - \epsilon \right], \quad (1.7)$$

for $\eta > 0$ small enough and

$$\bar{u}_\epsilon \stackrel{\text{def}}{=} M \left[\left(\phi_1 + \epsilon \frac{p-1+\delta}{p} \right)^{\frac{p}{p-1+\delta}} - \epsilon \right], \quad (1.8)$$

for $M > 0$ large enough. Since the operator $u \rightarrow -\Delta_p u - \frac{1}{(u+\epsilon)^\delta}$ is monotone from $(W_0^{1,p}(\Omega))^+$ to $W^{-1, \frac{p}{p-1}}(\Omega)$ (see Deimling [12] for further details about the theory of monotone operators), we get

from the weak comparison principle that

$$\underline{u}_\epsilon \leq u_\epsilon \leq \bar{u}_\epsilon. \tag{1.9}$$

Again from the weak comparison principle, we have that

$$0 < \epsilon_1 < \epsilon_2 \implies \begin{cases} u_{\epsilon_2} < u_{\epsilon_1} & \text{in } \Omega, \\ u_{\epsilon_1} + \epsilon_1 < u_{\epsilon_2} + \epsilon_2 & \text{in } \Omega, \end{cases}$$

from which it follows that $(u_{\epsilon_n})_{n \in \mathbb{N}}$ is a Cauchy sequence as $\epsilon_n \rightarrow 0^+$ in $C_0(\bar{\Omega})$. Then $u_{\epsilon_n} \rightarrow u$ in $C_0(\bar{\Omega})$ and by passing to the limit in (1.9) we deduce that $\underline{u} \leq u \leq \bar{u}$ where \underline{u} and \bar{u} are the respective subsolution and supersolution to (P) given by

$$\phi = \begin{cases} \phi_1 \left(\ln \left(\frac{A}{\phi_1} \right) \right)^{\frac{1}{p}} & \text{if } \delta = 1, \\ \phi_1^{\frac{p}{p-1+\delta}} & \text{if } \delta > 1, \end{cases} \quad \text{and} \quad \begin{cases} \underline{u} = \eta \phi, \\ \bar{u} = M \phi \end{cases} \tag{1.10}$$

(with $A, M > 0$ large enough and $\eta > 0$ small enough, depending on λ, g). Then it follows that $u \in \mathcal{C} \cap C_0(\bar{\Omega})$. Let us show that u is a weak solution to (P). Since $\delta < 2 + \frac{1}{p-1}$, we get from (1.9) and the Hardy Inequality that

$$\limsup_{n \in \mathbb{N}} \int_{\Omega} \frac{u_{\epsilon_n}}{(u_{\epsilon_n} + \epsilon_n)^\delta} dx < +\infty,$$

and consequently, by multiplying by u_{ϵ_n} the first equation of (P_{ϵ_n}) and integrating by parts, we obtain $\sup_{n \in \mathbb{N}} \|u_{\epsilon_n}\|_{W_0^{1,p}(\Omega)} < +\infty$. Moreover, by subtracting (P_{ϵ_n}) to (P_{ϵ_m}) and recalling the following well-known inequality (see [37]) for $p \geq 2$, w, v in $W^{1,p}(\Omega)$ and suitable $C_1 > 0$,

$$\int_{\Omega} (|\nabla w|^{p-2} \nabla w - |\nabla v|^{p-2} \nabla v) \nabla(w - v) dx \geq C_1 \int_{\Omega} |\nabla(w - v)|^p dx \tag{1.11}$$

and the following well-known inequality for $p < 2$, w, v in $W^{1,p}(\Omega)$ and suitable $C_2 > 0$,

$$\begin{aligned} & \int_{\Omega} (|\nabla w|^{p-2} \nabla w - |\nabla v|^{p-2} \nabla v) \nabla(w - v) dx \\ & \geq \frac{C_2 \left(\int_{\Omega} |\nabla(w - v)|^p dx \right)^{\frac{2}{p}}}{\left(\left(\int_{\Omega} |\nabla w|^p dx \right)^{\frac{1}{p}} + \left(\int_{\Omega} |\nabla v|^p dx \right)^{\frac{1}{p}} \right)^{2-p}}, \end{aligned} \tag{1.12}$$

we obtain

$$\langle -\Delta_p u_{\epsilon_n} + \Delta_p u_{\epsilon_m}, u_{\epsilon_n} - u_{\epsilon_m} \rangle \geq \begin{cases} C_1 \|u_{\epsilon_n} - u_{\epsilon_m}\|_{W_0^{1,p}(\Omega)}^p & \text{if } p \geq 2, \\ C_2 \frac{\|u_{\epsilon_n} - u_{\epsilon_m}\|_{W_0^{1,p}(\Omega)}^2}{\left(\|u_{\epsilon_n}\|_{W_0^{1,p}(\Omega)} + \|u_{\epsilon_m}\|_{W_0^{1,p}(\Omega)} \right)^{2-p}} & \text{if } p < 2. \end{cases}$$

Then we deduce that u_{ϵ_n} is also a Cauchy sequence in $W_0^{1,p}(\Omega)$ as $\epsilon_n \rightarrow 0^+$ and that $u_{\epsilon_n} \rightarrow u$ in $W_0^{1,p}(\Omega)$. Thus, it is easy to derive that u is a weak solution to (P). Finally, the uniqueness of

the solution to (P) in $W_0^{1,p}(\Omega) \cap \mathcal{C}$ follows from the strict monotonicity of $u \rightarrow u - \lambda(\Delta_p u - \frac{1}{u^\delta})$ from $W_0^{1,p}(\Omega) \cap \mathcal{C}$ to $W^{-1, \frac{p}{p-1}}(\Omega)$ which is a consequence of (1.11), (1.12). \square

We prove now Theorem 0.6.

Proof of Theorem 0.6. Let $\delta \geq 2 + \frac{1}{p-1}$. We give an alternative proof for existence of solutions. Let $(\Omega_k)_k$ be an increasing sequence of smooth domains such that $\Omega_k \uparrow \Omega$ (in the Hausdorff Topology) and $\frac{1}{k} \leq \text{dist}(x, \partial\Omega) \leq \frac{2}{k}, \forall x \in \Omega_k$. We use the subsolution and supersolution technique in Ω_k and pass to the limit as $k \rightarrow \infty$. For $0 < \eta < M$, let

$$\underline{u} \stackrel{\text{def}}{=} \eta(\phi_1)^{\frac{p}{p-1+\delta}}, \quad \bar{u} \stackrel{\text{def}}{=} M(\phi_1)^{\frac{p}{p-1+\delta}}.$$

For η small enough and M large enough, \underline{u} and \bar{u} are respectively a subsolution and a supersolution to (P) and both belong to $\mathcal{C} \cap C_0(\bar{\Omega})$. By using a minimization argument in $W_0^{1,p}(\Omega_k)$ as in the case $\delta < 1$ (note that the term associated to $\frac{1}{u^\delta}$ in the energy functional is not singular since $\underline{u} > 0$ on $\partial\Omega_k$), there is a positive solution $v_k \in W_0^{1,p}(\Omega_k)$ to

$$\begin{cases} u - \lambda \left(\Delta_p(u + \underline{u}) + \frac{1}{(u + \underline{u})^\delta} \right) = g - \underline{u} & \text{in } \Omega_k, \\ u = 0 & \text{on } \partial\Omega_k. \end{cases}$$

From Lieberman [27], $v_k \in C^{1,\beta}(\bar{\Omega}_k)$ for some $\beta \in (0, 1)$. Then, $u_k \stackrel{\text{def}}{=} u + \underline{u} \in C^{1,\beta}(\bar{\Omega}_k)$ satisfies

$$\begin{cases} u_k - \lambda \left(\Delta_p u_k + \frac{1}{u_k^\delta} \right) = g & \text{in } \Omega_k, \\ u = \underline{u} & \text{on } \partial\Omega_k, \end{cases}$$

and $\underline{u} \leq u_k \leq \bar{u}$ holds. From the weak comparison principle, we have that $u_k \leq u_{k+1}$ in Ω_k , and if $\tilde{u}_k \in C_0(\bar{\Omega})$ denotes the extension of u_k by \underline{u} outside Ω_k , then $\underline{u} \leq \tilde{u}_k \leq \tilde{u}_{k+1} \leq \bar{u}$ and by Dini's Theorem, $\tilde{u}_k \rightarrow u$ in $C_0(\bar{\Omega}) \cap \mathcal{C}$. Moreover, for every compact subset \mathcal{K} of Ω and k large enough so that $\mathcal{K} \subset \Omega_k$, we have $\frac{1}{\tilde{u}_k^\delta} = \frac{1}{u_k^\delta} \leq \frac{1}{\underline{u}^\delta} \in L^\infty(\mathcal{K})$ and $\Delta_p \tilde{u}_k = \Delta_p u_k = -g + u_k - \frac{\lambda}{u_k^\delta}$ bounded in $L^\infty(\mathcal{K})$ uniformly in k . Then using local regularity results (see for instance Serrin [36], Tolksdorf [40] and [39], DiBenedetto [16]), for k large enough we get that u_k is bounded in $C^1(\mathcal{K})$ and then converges to u in $W^{1,p}(\mathcal{K})$. Then $\underline{u} \leq u \stackrel{\text{def}}{=} \lim_{k \rightarrow \infty} u_k \in W_{\text{loc}}^{1,p}(\Omega)$ and satisfies (P) in the sense of distributions. Let us show that $u \notin W_0^{1,p}(\Omega)$. For that, we argue by contradiction: assume that $u \in W_0^{1,p}(\Omega)$. Then, from the equation in (P), we get that $\frac{1}{u^\delta} \in W^{-1, \frac{p}{p-1}}(\Omega)$. Thus, $\int_\Omega \bar{u}^{1-\delta} dx \leq \int_\Omega u^{1-\delta} dx < +\infty$ which contradicts the definition of \bar{u} . The proof of Theorem 0.6 is now complete. \square

2. Proof of Theorem 0.9 and Proposition 0.1

Using Theorem 0.5 and a semi-discretization in time with implicit Euler method, we prove Theorem 0.9.

Proof of Theorem 0.9. Let $N \in \mathbb{N}^*, n \geq 2$ and $\Delta_t = \frac{T}{N}$. For $0 \leq n \leq N$, we define $t_n \stackrel{\text{def}}{=} n\Delta_t$, $h^n(\cdot) \stackrel{\text{def}}{=} \frac{1}{\Delta_t} \int_{t_{n-1}}^{t_n} h(\tau, \cdot) d\tau \in L^\infty(\Omega)$ and the function $h_{\Delta_t} \in L^\infty(Q_T)$ as follows

$$h_{\Delta_t}(t) \stackrel{\text{def}}{=} h^n, \quad \forall t \in [t_{n-1}, t_n), \quad \forall n \in \{1, \dots, N\}.$$

Notice that we have for all $1 < q < +\infty$:

$$\|h_{\Delta_t}\|_{L^q(Q_T)} \leq (T|\Omega|)^{\frac{1}{q}} \|h\|_{\infty}, \tag{2.1}$$

$$h_{\Delta_t} \rightarrow h \text{ in } L^q(Q_T). \tag{2.2}$$

From Theorem 0.5 (with $\lambda = \Delta_t$, $g = \Delta_t h^n + u^{n-1} \in L^\infty(\Omega)$), we define by iteration $u^n \in W_0^{1,p}(\Omega) \cap \mathcal{C}$ with the following implicit Euler scheme:

$$\begin{cases} \frac{u^n - u^{n-1}}{\Delta_t} - \Delta_p u^n - \frac{1}{(u^n)^\delta} = h^n & \text{in } \Omega, \\ u^n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

and $u^0 = u_0 \in W_0^{1,p}(\Omega) \cap \mathcal{C}$. Then, defining functions $u_{\Delta_t}, \tilde{u}_{\Delta_t}$ by: for all $n \in \{1, \dots, N\}$,

$$\forall t \in [t_{n-1}, t_n), \quad \begin{cases} u_{\Delta_t}(t) \stackrel{\text{def}}{=} u^n, \\ \tilde{u}_{\Delta_t}(t) \stackrel{\text{def}}{=} \frac{(t - t_{n-1})}{\Delta_t} (u^n - u^{n-1}) + u^{n-1}, \end{cases} \tag{2.4}$$

we have that

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} - \Delta_p u_{\Delta_t} - \frac{1}{u_{\Delta_t}^\delta} = h_{\Delta_t} \in L^\infty(Q_T). \tag{2.5}$$

Using energy estimates, we first establish some a priori estimates for u_{Δ_t} and \tilde{u}_{Δ_t} independent of Δ_t . Precisely, multiplying (2.3) by $\Delta_t u^n$ and summing from $n = 1$ to $N' \leq N$, we get for $\epsilon > 0$ small, by the Young Inequality and (2.1),

$$\begin{aligned} & \sum_{n=1}^{N'} \int_{\Omega} (u^n - u^{n-1}) u^n \, dx + \Delta_t \left[\sum_{n=1}^{N'} \|u^n\|_{W_0^{1,p}(\Omega)}^p - \sum_{n=1}^{N'} \int_{\Omega} (u^n)^{1-\delta} \, dx \right] \\ & \leq C(\epsilon) T |\Omega| \|h\|_{\infty}^{\frac{p}{p-1}} + \epsilon \Delta_t \sum_{n=1}^{N'} \|u^n\|_{W_0^{1,p}(\Omega)}^p. \end{aligned} \tag{2.6}$$

In addition,

$$\begin{aligned} \sum_{n=1}^{N'} \int_{\Omega} (u^n - u^{n-1}) u^n \, dx &= \frac{1}{2} \sum_{n=1}^{N'} \int_{\Omega} (|u^n - u^{n-1}|^2 + |u^n|^2 - |u^{n-1}|^2) \, dx \\ &= \frac{1}{2} \sum_{n=1}^{N'} \int_{\Omega} |u^n - u^{n-1}|^2 \, dx + \frac{1}{2} \int_{\Omega} |u^{N'}|^2 \, dx - \frac{1}{2} \int_{\Omega} |u_0|^2 \, dx. \end{aligned} \tag{2.7}$$

Next, we estimate the singular term in the above expression. For that, arguing as in the proof of Theorem 0.5, we can prove the existence of $\underline{u}, \bar{u} \in W_0^{1,p}(\Omega) \cap \mathcal{C}$ such that $\underline{u} \leq u_0 \leq \bar{u}$ (since $u_0 \in \mathcal{C}$) and such that

$$\begin{aligned}
 -\Delta_p \underline{u} - \frac{1}{\underline{u}^\delta} &\leq -\|h\|_{L^\infty(Q_T)} \quad \text{in } \Omega, \\
 -\Delta_p \bar{u} - \frac{1}{\bar{u}^\delta} &\geq \|h\|_{L^\infty(Q_T)} \quad \text{in } \Omega.
 \end{aligned}$$

Indeed, if $\delta < 1$ choose $\underline{u} = \eta\phi_1$ and $\bar{u} = MU$ with U solution of (1.4) and if $\delta \geq 1$ choose \underline{u}, \bar{u} as in (1.10), where $A > 0, M > 0$ are large enough and $\eta > 0$ is small enough. Note that A, M, η depend on $\|h\|_{L^\infty(Q_T)}$. Then iterating the application of the weak comparison principle, we obtain that for all $n \in \mathbb{N}, \underline{u} \leq u^n \leq \bar{u}$ which implies that

$$\underline{u} \leq u_{\Delta t}, \tilde{u}_{\Delta t} \leq \bar{u}. \tag{2.8}$$

Therefore, since $\delta < 2 + \frac{1}{p-1}$,

$$\Delta_t \sum_{n=1}^{N'} \int_{\Omega} (u^n)^{1-\delta} dx \leq \begin{cases} T \int_{\Omega} \bar{u}^{1-\delta} dx < +\infty & \text{if } \delta \leq 1, \\ T \int_{\Omega} \underline{u}^{1-\delta} dx < +\infty & \text{if } \delta > 1. \end{cases} \tag{2.9}$$

Gathering the (2.6), (2.7), (2.8) and (2.9), we get that $u_{\Delta t}, \tilde{u}_{\Delta t} \in \mathcal{C}$ uniformly and are bounded in $L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega))$. We now use a second energy estimate. Multiplying (2.3) by $u^n - u^{n-1}$ and summing from $n = 1$ to $N' \leq N$, we get by the Young Inequality

$$\begin{aligned}
 &\Delta_t \sum_{n=1}^{N'} \int_{\Omega} \left(\frac{u^n - u^{n-1}}{\Delta t} \right)^2 dx + \sum_{n=1}^{N'} \int_{\Omega} |\nabla u^n|^{p-2} \nabla u^n \cdot \nabla (u^n - u^{n-1}) dx - \sum_{n=1}^{N'} \int_{\Omega} \frac{u^n - u^{n-1}}{(u^n)^\delta} dx \\
 &\leq \frac{\Delta t}{2} \sum_{n=1}^{N'} \left[\int_{\Omega} (h^n)^2 dx + \int_{\Omega} \left(\frac{u^n - u^{n-1}}{\Delta t} \right)^2 dx \right]
 \end{aligned} \tag{2.10}$$

which implies that

$$\begin{aligned}
 &\frac{\Delta t}{2} \sum_{n=1}^{N'} \int_{\Omega} \left(\frac{u^n - u^{n-1}}{\Delta t} \right)^2 dx + \sum_{n=1}^{N'} \int_{\Omega} |\nabla u^n|^{p-2} \nabla u^n \cdot \nabla (u^n - u^{n-1}) dx - \sum_{n=1}^{N'} \int_{\Omega} \frac{u^n - u^{n-1}}{(u^n)^\delta} dx \\
 &\leq |\Omega| \frac{T}{2} \|h\|_{L^\infty(Q_T)}^2.
 \end{aligned} \tag{2.11}$$

From the convexity of the terms $\int_{\Omega} |\nabla u|^p dx$ and $-\frac{1}{1-\delta} \int_{\Omega} u^{1-\delta} dx$ we derive the following estimates:

$$\begin{aligned}
 &\frac{1}{p} \left[\int_{\Omega} |\nabla u^n|^p dx - \int_{\Omega} |\nabla u^{n-1}|^p dx \right] \leq \int_{\Omega} |\nabla u^n|^{p-2} \nabla u^n \nabla (u^n - u^{n-1}) dx, \\
 &\frac{1}{1-\delta} \left[\int_{\Omega} (u^{n-1})^{1-\delta} dx - \int_{\Omega} (u^n)^{1-\delta} dx \right] \leq - \int_{\Omega} \frac{u^n - u^{n-1}}{(u^n)^\delta} dx.
 \end{aligned} \tag{2.12}$$

Therefore, gathering the estimates (2.11) and (2.12), we get

$$\begin{aligned} & \frac{\Delta_t}{2} \sum_{n=1}^{N'} \int_{\Omega} \left(\frac{u^n - u^{n-1}}{\Delta_t} \right)^2 dx + \frac{1}{p} \left[\int_{\Omega} |\nabla u^{N'}|^p dx - \int_{\Omega} |\nabla u_0|^p dx \right] \\ & + \frac{1}{1-\delta} \left[\int_{\Omega} (u_0)^{1-\delta} dx - \int_{\Omega} (u^{N'})^{1-\delta} dx \right] \\ & \leq |\Omega| \frac{T}{2} \|h\|_{L^\infty(Q_T)}^2. \end{aligned} \tag{2.13}$$

The above expression together with $\int_{\Omega} (u^n)^{1-\delta} dx \leq \max\{\int_{\Omega} (\bar{u})^{1-\delta} dx, \int_{\Omega} (\underline{u})^{1-\delta} dx\}$ yields

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} \text{ is bounded in } L^2(Q_T) \text{ uniformly in } \Delta_t, \tag{2.14}$$

$$u_{\Delta_t}, \tilde{u}_{\Delta_t} \text{ are bounded in } L^\infty(0, T; W_0^{1,p}(\Omega)) \text{ uniformly in } \Delta_t. \tag{2.15}$$

Furthermore, from above there exists $C > 0$ independent of Δ_t such that

$$\|u_{\Delta_t} - \tilde{u}_{\Delta_t}\|_{L^\infty(0,T;L^2(\Omega))} \leq \max_{n \in \{1, \dots, N\}} \|u^n - u^{n-1}\|_{L^2(\Omega)} \leq C(\Delta_t)^{\frac{1}{2}}. \tag{2.16}$$

Therefore, taking $N \rightarrow \infty$ (which implies that $\Delta_t \rightarrow 0^+$), and up to a subsequence, we get from (2.14) and (2.15) that there exist $u, v \in L^\infty(0, T; W_0^{1,p}(\Omega) \cap L^\infty(\Omega))$ such that $\frac{\partial u}{\partial t} \in L^2(Q_T)$, $u, v \in C$ uniformly and as $\Delta_t \rightarrow 0^+$,

$$\tilde{u}_{\Delta_t} \xrightarrow{*} u \text{ in } L^\infty(0, T; W_0^{1,p}(\Omega) \cap L^\infty(\Omega)), \tag{2.17}$$

$$u_{\Delta_t} \xrightarrow{*} v \text{ in } L^\infty(0, T; W_0^{1,p}(\Omega) \cap L^\infty(\Omega)), \tag{2.18}$$

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \text{ in } L^2(Q_T). \tag{2.19}$$

From (2.16), it follows that $u \equiv v$. Moreover, from (2.8), it follows that $\underline{u} \leq u \leq \bar{u}$. Therefore, $u \in \mathbf{V}(Q_T)$.

Next, let us prove that u satisfies (in the sense of Definition 0.2) the first equation in (S_t) . Using the boundedness of $\frac{\partial \tilde{u}_{\Delta_t}}{\partial t}$ in $L^2(Q_T)$ given by (2.14), we first get that $\{\tilde{u}_{\Delta_t}\}_{\Delta_t}$ is equicontinuous in $C(0, T; L^q(\Omega))$ for $1 \leq q \leq 2$, and thus with $\underline{u} \leq \tilde{u}_{\Delta_t} \leq \bar{u}$ and the interpolation inequality $\|\cdot\|_r \leq \|\cdot\|_\infty^\alpha \|\cdot\|_2^{1-\alpha}$, $\frac{1}{r} = \frac{\alpha}{\infty} + \frac{1-\alpha}{2}$, we obtain that $\{\tilde{u}_{\Delta_t}\}_{\Delta_t}$ is equicontinuous in $C(0, T; L^q(\Omega))$ for any $1 < q < +\infty$. Moreover, since $\{\tilde{u}_{\Delta_t}\}_{\Delta_t}$ is a bounded family of $W_0^{1,p}(\Omega)$ which is compactly embedded in $L^q(\Omega)$ for $1 < q < \frac{Np}{N-p}$, and from Ascoli–Arzela Theorem, and using again the interpolation inequality, we get as $\Delta_t \rightarrow 0^+$ that up to a subsequence

$$\tilde{u}_{\Delta_t} \rightarrow u \text{ in } C(0, T; L^q(\Omega)), \forall q > 1, \tag{2.20}$$

and then, from (2.16) (with the interpolation inequality for $q > 2$), it follows that

$$u_{\Delta_t} \rightarrow u \text{ in } L^\infty(0, T; L^q(\Omega)), \forall q > 1 \tag{2.21}$$

as $\Delta_t \rightarrow 0^+$. Thus, multiplying (2.5) by $(u_{\Delta_t} - u)$ and using (2.20)–(2.21), we get by straightforward calculations:

$$\begin{aligned} & \int_0^T \int_{\Omega} \left[\frac{\partial \tilde{u}_{\Delta t}}{\partial t} - \frac{\partial u}{\partial t} \right] (\tilde{u}_{\Delta t} - u) \, dx \, dt - \int_0^T \langle \Delta_p u_{\Delta t}, u_{\Delta t} - u \rangle \, dt - \int_0^T \int_{\Omega} u_{\Delta t}^{-\delta} (u_{\Delta t} - u) \, dx \, dt \\ &= \int_0^T \int_{\Omega} h_{\Delta t} (u_{\Delta t} - u) \, dx \, dt + o_{\Delta t}(1). \end{aligned}$$

From (2.8) and (2.20), we have that

$$\int_0^T \int_{\Omega} u_{\Delta t}^{-\delta} (u_{\Delta t} - u) \, dx \, dt = o_{\Delta t}(1),$$

and from (2.1) and (2.20) we have

$$\int_0^T \int_{\Omega} h_{\Delta t} (u_{\Delta t} - u) \, dx \, dt = o_{\Delta t}(1).$$

Then,

$$\frac{1}{2} \int_{\Omega} |\tilde{u}_{\Delta t} - u|^2(T) \, dx - \int_0^T \langle \Delta_p u_{\Delta t} - \Delta_p u, u_{\Delta t} - u \rangle \, dt = o_{\Delta t}(1).$$

Therefore, using (2.21), $u \neq 0$ and the inequality (1.11) with $w = u_{\Delta t}(t)$ and $v = u(t)$ we obtain that $u_{\Delta t} \rightarrow u$ in $L^p(0, T; W_0^{1,p}(\Omega))$ and

$$-\Delta_p u_{\Delta t} \rightarrow -\Delta_p u \quad \text{in } L^{\frac{p}{p-1}}(0, T; W^{-1, \frac{p}{p-1}}(\Omega)). \tag{2.22}$$

Moreover, from (2.8), for any $\phi \in W_0^{1,p}(\Omega)$

$$\left| \int_{\Omega} \frac{\phi}{(u_{\Delta t})^{\delta}} \, dx \right| \leq \int_{\Omega} \frac{|\phi|}{(\underline{u})^{\delta}} \, dx \leq \left(\int_{\Omega} \left(\frac{d(x)}{(\underline{u})^{\delta}} \right)^{\frac{p}{p-1}} \, dx \right)^{\frac{p-1}{p}} \times \left(\int_{\Omega} \left(\frac{|\phi|}{d(x)} \right)^p \, dx \right)^{\frac{1}{p}}$$

and since $\delta < 2 + \frac{1}{p-1}$

$$\int_{\Omega} \left(\frac{d(x)}{(\underline{u})^{\delta}} \right)^{\frac{p}{p-1}} \, dx < +\infty.$$

Then, from the Hardy Inequality and from the Lebesgue Theorem, we obtain

$$\frac{1}{(u_{\Delta t})^{\delta}} \rightarrow \frac{1}{u^{\delta}} \quad \text{in } L^{\infty}(0, T; W^{-1, \frac{p}{p-1}}(\Omega)). \tag{2.23}$$

Therefore, from (2.2), (2.20), (2.21), (2.22), (2.23) we deduce that $u \in \mathbf{V}(Q_T)$ satisfies (P_t) .

Let us now show that u is the unique weak solution such that $u(t) \in C, \forall t \in [0, T]$. Assume that there exists $v \neq u$ a weak solution to (P_t) satisfying $v(t) \in C, \forall t \in [0, T]$. Then,

$$\int_0^T \int_{\Omega} \frac{\partial(u-v)}{\partial t} (u-v) \, dx \, dt - \int_0^T \langle \Delta_p u - \Delta_p v, u-v \rangle \, dt - \int_0^T \int_{\Omega} (u^{-\delta} - v^{-\delta})(u-v) \, dx \, dt = 0.$$

The above equality together with $u(0) = v(0)$ imply $u \equiv v$.

To complete the proof of Theorem 0.9, let us prove $u \in C([0, T]; W_0^{1,p}(\Omega))$ and (0.3). First, we observe that since $u \in C([0, T], L^2(\Omega))$ and $u \in L^\infty(0, T; W_0^{1,p}(\Omega))$, it follows that $u : t \in [0, T] \rightarrow W_0^{1,p}(\Omega)$ is weakly continuous and then that $u(t_0) \in W_0^{1,p}(\Omega)$ and $\|u(t_0)\|_{W_0^{1,p}(\Omega)} \leq \liminf_{t \rightarrow t_0} \|u(t)\|_{W_0^{1,p}(\Omega)}$ for all $t_0 \in [0, T]$. From (2.2), (2.13) (with $\sum_{n=N''}^{N'}$ for $1 \leq N'' \leq N'$ instead of $\sum_{n=1}^{N'}$), (2.16) and (2.23), it follows that u satisfies for any $t \in [t_0, T]$:

$$\begin{aligned} & \int_{t_0}^t \int_{\Omega} \left(\frac{\partial u}{\partial t}\right)^2 \, dx \, ds + \frac{1}{p} \int_{\Omega} |\nabla u(t)|^p \, dx - \frac{1}{1-\delta} \int_{\Omega} u(t)^{1-\delta} \, dx \\ & \leq \int_{t_0}^t \int_{\Omega} h \frac{\partial u}{\partial t} \, dx \, ds + \frac{1}{p} \int_{\Omega} |\nabla u(t_0)|^p \, dx - \frac{1}{1-\delta} \int_{\Omega} u(t_0)^{1-\delta} \, dx. \end{aligned} \tag{2.24}$$

From (2.24) and Lebesgue Theorem, it follows that

$$\limsup_{t \rightarrow t_0^+} \|u(t)\|_{W_0^{1,p}(\Omega)} \leq \|u(t_0)\|_{W_0^{1,p}(\Omega)}$$

and then $u(t) \rightarrow u(t_0)$ in $W_0^{1,p}(\Omega)$ as $t \rightarrow t_0^+$ which implies that u is right-continuous on $[0, T]$. Let $t > t_0$. Let us now prove the left-continuity. Let $0 < k \leq t - t_0$. Multiplying (0.2) by $\tau_k(u)(s) \stackrel{\text{def}}{=} \frac{u(s+k) - u(s)}{k}$ and integrating over $(t_0, t) \times \Omega$, we get, using convexity arguments, that

$$\begin{aligned} & \int_{t_0}^t \int_{\Omega} \frac{\partial u}{\partial t} \tau_k(u) \, dx \, ds + \frac{1}{kp} \int_{t_0}^t \int_{\Omega} (|\nabla u(s+k)|^p - |\nabla u(s)|^p) \, dx \, ds \\ & - \frac{1}{(1-\delta)k} \int_{t_0}^t \int_{\Omega} (u^{1-\delta}(s+k) - u^{1-\delta}(s)) \, dx \, ds \\ & \geq \int_{t_0}^t \int_{\Omega} h \tau_k(u) \, dx \, ds. \end{aligned}$$

Then,

$$\begin{aligned} & \int_{t_0}^t \int_{\Omega} \frac{\partial u}{\partial t} \tau_k(u) \, dx \, ds + \frac{1}{kp} \left(\int_t^{t+k} \int_{\Omega} |\nabla u(s)|^p \, dx \, ds - \int_{t_0}^{t_0+k} \int_{\Omega} |\nabla u(s)|^p \, dx \, ds \right) \\ & - \frac{1}{(1-\delta)k} \left(\int_t^{t+k} \int_{\Omega} \left(u^{1-\delta}(s) \, dx \, ds - \int_{t_0}^{t_0+k} u^{1-\delta}(s) \, dx \, ds \right) \right) \\ & \geq \int_{t_0}^t \int_{\Omega} h \tau_k(u) \, dx \, ds. \end{aligned}$$

Since u is right-continuous in $W_0^{1,p}(\Omega)$ and by Lebesgue Theorem, we get as $k \rightarrow 0^+$

$$\begin{aligned} \frac{1}{k} \int_t^{t+k} \int_{\Omega} |\nabla u(s)|^p \, dx \, ds &\rightarrow \int_{\Omega} |\nabla u(t)|^p \, dx, \\ \frac{1}{k} \int_{t_0}^{t_0+k} \int_{\Omega} |\nabla u(s)|^p \, dx \, ds &\rightarrow \int_{\Omega} |\nabla u(t_0)|^p \, dx, \\ \frac{1}{k} \int_t^{t+k} \int_{\Omega} u^{1-\delta}(s) \, dx \, ds &\rightarrow \int_{\Omega} u^{1-\delta}(t) \, dx, \\ \frac{1}{k} \int_{t_0}^{t_0+k} \int_{\Omega} u^{1-\delta}(s) \, dx \, ds &\rightarrow \int_{\Omega} u(t_0)^{1-\delta} \, dx. \end{aligned}$$

From the above estimates, we get as $k \rightarrow 0^+$

$$\begin{aligned} & \int_{t_0}^t \int_{\Omega} \left(\frac{\partial u}{\partial t} \right)^2 \, dx \, ds + \frac{1}{p} \int_{\Omega} |\nabla u(t)|^p \, dx - \frac{1}{1-\delta} \int_{\Omega} u(t)^{1-\delta} \, dx \\ & \geq \int_{t_0}^t \int_{\Omega} h \frac{\partial u}{\partial t} \, dx \, ds + \frac{1}{p} \int_{\Omega} |\nabla u(t_0)|^p \, dx - \frac{1}{1-\delta} \int_{\Omega} u(t_0)^{1-\delta} \, dx, \end{aligned}$$

which implies together with (2.24) that the above inequality is in fact an equality. Then together with the fact that $t \rightarrow \int_{\Omega} u^{1-\delta}(t) \, dx$ is continuous, it follows that $u \in C([0, T], W_0^{1,p}(\Omega))$. Finally (0.3) is obtained by setting $t_0 = 0$. \square

We end this section by proving Proposition 0.1.

Proof of Proposition 0.1. Assume that $u_0 \in \overline{\mathcal{D}(A)}^{L^\infty(\Omega)}$, where A and $\mathcal{D}(A)$ are defined in (0.4). From (1.3), A is m -accretive in $L^\infty(\Omega)$. Indeed, for $f, g \in L^\infty(\Omega)$ and $\lambda > 0$, set u and $v \in \mathcal{D}(A)$ (given by Theorem 0.5) satisfying

$$\begin{aligned} u - \lambda Au &= f \quad \text{in } \Omega, \\ v - \lambda Av &= g \quad \text{in } \Omega. \end{aligned} \tag{2.25}$$

From (2.25) and defining $w \stackrel{\text{def}}{=} (u - v - \|f - g\|_{L^\infty(\Omega)})^+$, we get that

$$\int_{\Omega} w^2 \, dx + \lambda \langle Au - Av, w \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} \leq 0.$$

From (1.11) or (1.12) it follows that $u - v \leq \|f - g\|_{L^\infty(\Omega)}$ and reversing the roles of u and v , we get that $\|u - v\|_{L^\infty(\Omega)} \leq \|f - g\|_{L^\infty(\Omega)}$. Then Proposition 0.1 can be obtained from [6, Chap. 4, Thm. 4.2 and Thm. 4.4]. However, in order to be complete and self-contained, let us briefly explain the argument. In the following, $\|\cdot\|_\infty$ stands for the norm of $L^\infty(\Omega)$. For $z \in \mathcal{D}(A)$ and r, k in $L^\infty(Q_T)$ let define

$$\varphi(t, s) = \|r(t) - k(s)\|_\infty, \quad (t, s) \in [0, T] \times [0, T],$$

and

$$\begin{aligned} b(t, r, k) &= \|u_0 - z\|_\infty + \|v_0 - z\|_\infty + |t| \|Az\|_\infty \\ &\quad + \int_0^{t^+} \|r(\tau)\|_\infty \, d\tau + \int_0^{t^-} \|k(\tau)\|_\infty \, d\tau, \quad t \in [-T, T], \end{aligned}$$

and

$$\Psi(t, s) = b(t - s, r, k) + \begin{cases} \int_0^s \varphi(t - s + \tau, \tau) \, d\tau & \text{if } 0 \leq s \leq t \leq T, \\ \int_0^t \varphi(\tau, s - t + \tau) \, d\tau & \text{if } 0 \leq t \leq s \leq T, \end{cases}$$

the solution of

$$\begin{cases} \frac{\partial \Psi}{\partial t}(t, s) + \frac{\partial \Psi}{\partial s}(t, s) = \varphi(t, s), & (t, s) \in [0, T] \times [0, T], \\ \Psi(t, 0) = b(t, r, k), & t \in [0, T], \\ \Psi(0, s) = b(-s, r, k), & s \in [0, T]. \end{cases} \tag{2.26}$$

Moreover, let denote by (u_ϵ^n) the solution of (2.3) with $\Delta_t = \epsilon$, $h = r$, $r^n = \frac{1}{\epsilon} \int_{(n-1)\epsilon}^{n\epsilon} r(\tau, \cdot) \, d\tau$ and (u_η^n) the solution of (2.3) with $\Delta_t = \eta$, $h = k$, $k^n = \frac{1}{\eta} \int_{(n-1)\eta}^{n\eta} k(\tau, \cdot) \, d\tau$ respectively. For $(n, m) \in \mathbb{N}^*$ elementary calculations lead to

$$u_\epsilon^n - u_\eta^m + \frac{\epsilon \eta}{\epsilon + \eta} (Au_\epsilon^n - Au_\eta^m) = \frac{\eta}{\epsilon + \eta} (u_\epsilon^{n-1} - u_\eta^m) + \frac{\epsilon}{\epsilon + \eta} (u_\epsilon^n - u_\eta^{m-1}) + \frac{\epsilon \eta}{\epsilon + \eta} (r^n - k^m),$$

and since A is m -accretive in $L^\infty(\Omega)$ we first verify that $\Phi_{n,m}^{\epsilon,\eta} = \|u_\epsilon^n - u_\eta^m\|_\infty$ obeys

$$\begin{aligned} \Phi_{n,m}^{\epsilon,\eta} &\leq \frac{\eta}{\epsilon + \eta} \Phi_{n-1,m}^{\epsilon,\eta} + \frac{\epsilon}{\epsilon + \eta} \Phi_{n,m-1}^{\epsilon,\eta} + \frac{\epsilon \eta}{\epsilon + \eta} \|r^n - k^m\|_\infty, \\ \Phi_{n,0}^{\epsilon,\eta} &\leq b(t_n, r_\epsilon, k_\eta) \quad \text{and} \quad \Phi_{0,m}^{\epsilon,\eta} \leq b(-s_m, r_\epsilon, k_\eta), \end{aligned}$$

and thus, with an easy inductive argument, that $\Phi_{n,m}^{\epsilon,\eta} \leq \Psi_{n,m}^{\epsilon,\eta}$ where $\Psi_{n,m}^{\epsilon,\eta}$ satisfies

$$\Psi_{n,m}^{\epsilon,\eta} = \frac{\eta}{\epsilon + \eta} \Psi_{n-1,m}^{\epsilon,\eta} + \frac{\epsilon}{\epsilon + \eta} \Psi_{n,m-1}^{\epsilon,\eta} + \frac{\epsilon\eta}{\epsilon + \eta} \|h_\epsilon^n - h_\eta^m\|_\infty,$$

$$\Psi_{n,0}^{\epsilon,\eta} = b(t_n, r_\epsilon, k_\eta) \quad \text{and} \quad \Psi_{0,m}^{\epsilon,\eta} = b(-s_m, r_\epsilon, k_\eta).$$

For $(t, s) \in (t_{n-1}, t_n) \times (s_{m-1}, s_m)$ let set $\varphi^{\epsilon,\eta}(t, s) = \|r_\epsilon(t) - k_\eta(s)\|_\infty$ and $\Psi^{\epsilon,\eta}(t, s) = \Psi_{n,m}^{\epsilon,\eta}$, $b_{\epsilon,\eta}(t, r, k) = b(t_n, r_\epsilon, k_\eta)$ and $b_{\epsilon,\eta}(-s, r, k) = b(-s_m, r_\epsilon, k_\eta)$. Then $\Psi^{\epsilon,\eta}$ satisfies the following discrete version of (2.26):

$$\frac{\Psi^{\epsilon,\eta}(t, s) - \Psi^{\epsilon,\eta}(t - \epsilon, s)}{\epsilon} + \frac{\Psi^{\epsilon,\eta}(t, s) - \Psi^{\epsilon,\eta}(t, s - \eta)}{\eta} = \varphi^{\epsilon,\eta}(t, s),$$

$$\Psi^{\epsilon,\eta}(t, 0) = b_{\epsilon,\eta}(t, r, k) \quad \text{and} \quad \Psi^{\epsilon,\eta}(0, s) = b_{\epsilon,\eta}(s, r, k),$$

and from $b_{\epsilon,\eta}(\cdot, r, k) \rightarrow b(\cdot, r, k)$ in $L^\infty([0, T])$ and $\varphi^{\epsilon,\eta} \rightarrow \varphi$ in $L^\infty([0, T] \times [0, T])$ we deduce that $\rho_{\epsilon,\eta} = \|\Psi^{\epsilon,\eta} - \Psi\|_{L^\infty([0,T] \times [0,T])} \rightarrow 0$ as $(\epsilon, \eta) \rightarrow 0$ (see for instance [6, Chap. 4, Lem. 4.3]). Then from

$$\|u_\epsilon(t) - u_\eta(s)\|_\infty = \Phi^{\epsilon,\eta}(t, s) \leq \Psi^{\epsilon,\eta}(t, s) \leq \Psi(t, s) + \rho_{\epsilon,\eta}, \tag{2.27}$$

we obtain with $t = s, r = k = h, v_0 = u_0$:

$$\|u_\epsilon(t) - u_\eta(t)\|_\infty \leq 2\|u_0 - z\|_\infty + \rho_{\epsilon,\eta},$$

and since z can be chosen in $\mathcal{D}(A)$ arbitrary close to u_0 , we deduce that u_ϵ is a Cauchy sequence in $L^\infty(Q_T)$ and then that $u_\epsilon \rightarrow u$ in $L^\infty(Q_T)$. Thus, passing to the limit in (2.27) with $r = k = h, v_0 = u_0$ we obtain

$$\begin{aligned} \|u(t) - u(s)\|_\infty &\leq 2\|u_0 - z\|_\infty + |t - s|\|Az\|_\infty + \int_0^{|t-s|} \|h(\tau)\|_\infty \, d\tau \\ &\quad + \int_0^{\max(t,s)} \|h(|t - s| + \tau) - h(\tau)\|_\infty \, d\tau, \end{aligned} \tag{2.28}$$

which, together with the density $\mathcal{D}(A)$ in $L^\infty(\Omega)$ and $h \in L^1(0, T; L^\infty(\Omega))$, yields $u \in C([0, T]; L^\infty(\Omega))$. Analogously, from (2.27) with $\epsilon = \eta = \Delta_t, r = k = h, v_0 = u_0$ and $t = s + \Delta_t$ we deduce that

$$\begin{aligned} \|u_{\Delta_t}(t) - \tilde{u}_{\Delta_t}(t)\|_\infty &\leq 2\|u_{\Delta_t}(t) - u_{\Delta_t}(t - \Delta_t)\|_\infty \\ &\leq 4\|u_0 - z\|_\infty + 2\Delta_t\|Az\|_\infty + 2\int_0^{\Delta_t} \|h(\tau)\|_\infty \, d\tau \\ &\quad + 2\int_0^t \|h(\Delta_t + \tau) - h(\tau)\|_\infty \, d\tau, \end{aligned}$$

which gives the limit $\tilde{u}_{\Delta_t} \rightarrow u$ in $C([0, T]; L^\infty(\Omega))$ as $\Delta_t \rightarrow 0^+$. Note that since $\tilde{u}_{\Delta_t} \in C([0, T]; C_0(\overline{\Omega}))$, the uniform limit u belongs to $C([0, T]; C_0(\overline{\Omega}))$. Moreover, passing to the limit in (2.27) with $t = s$ we obtain

$$\|u(t) - v(t)\|_\infty \leq \|u_0 - z\|_\infty + \|v_0 - z\|_\infty + \int_0^t \|r(\tau) - k(\tau)\|_\infty \, d\tau,$$

and (0.5) follows because we can choose z arbitrary close to v_0 . Finally, if $Au_0 \in L^\infty(\Omega)$ and $h \in W^{1,1}(0, T; L^\infty(\Omega))$ and if we assume (without loss of generality) that $t > s$ then with $z = v_0 = u(t - s)$ and $(r, k) = (h, h(\cdot + t - s))$ in the last above inequality we obtain

$$\begin{aligned} \|u(t) - u(s)\|_\infty &\leq \|u_0 - u(t - s)\|_\infty + \int_0^s \|h(\tau) - h(\tau + t - s)\|_\infty \, d\tau \\ &\leq \int_0^{t-s} \|Au_0 - h(\tau)\|_\infty \, d\tau + \int_0^s \|h(\tau) - h(\tau + t - s)\|_\infty \, d\tau \\ &\leq (t - s) \|Au_0 - h(0)\|_\infty + \int_0^{t-s} \|h(0) - h(\tau)\|_\infty \, d\tau + \int_0^s \|h(\tau) - h(\tau + t - s)\|_\infty \, d\tau \\ &\leq (t - s) \left(\|Au_0 - h(0)\|_\infty + \int_0^T \left\| \frac{dh(\tau)}{d\tau} \right\|_\infty \, d\tau \right). \end{aligned} \tag{2.29}$$

Note that the second above inequality is obtained from (0.5) with $v = u_0, k = Au_0$ and the last above inequality is obtained from $h(\tau) - h(\tau + t - s) = \int_\tau^{\tau+t-s} \frac{dh(\sigma)}{d\sigma} \, d\sigma$ together with Fubini's Theorem. Dividing the expression (2.29) by $|t - s|$, we get that u is a Lipschitz function and since $\frac{\partial u}{\partial t} \in L^2(Q_T)$, passing to the limit $|t - s| \rightarrow 0$ we obtain that $\frac{u(t) - u(s)}{t - s} \rightarrow \frac{\partial u}{\partial t}$ as $s \rightarrow t$ weakly in $L^2(Q_T)$ and $*$ -weakly in $L^\infty(Q_T)$. Furthermore,

$$\left\| \frac{\partial u}{\partial t} \right\|_\infty \leq \liminf_{s \rightarrow t} \frac{\|u(t) - u(s)\|_\infty}{|t - s|}.$$

Therefore, we get $u \in W^{1,\infty}(0, T; L^\infty(\Omega))$ as well as inequality (0.6). \square

3. Proofs of Theorems 0.8, 0.13, 0.15 and of Proposition 0.2

We start this section by the proof of Theorem 0.8. We use the following preliminary result which gives the validity of the weak comparison principle for sub-homogeneous problems and then forces uniqueness of solutions.

Theorem 3.1. *Let $1 < r < +\infty, g : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}$ be a Caratheodory function bounded below such that $\frac{g(x,s)}{s^{r-1}}$ is a decreasing function in \mathbb{R}^+ for a.e. $x \in \Omega$. Let $u \in L^\infty(\Omega) \cap W_0^{1,r}(\Omega), v \in L^\infty(\Omega) \cap W_0^{1,r}(\Omega)$ satisfy $u > 0, v > 0$ in $\Omega, \int_\Omega u^{1-\delta} \, dx < +\infty, \int_\Omega v^{1-\delta} \, dx < +\infty$ and*

$$-\Delta_r u \leq \frac{1}{u^\delta} + g(x, u) \quad \text{weakly in } W^{-1, \frac{r}{r-1}}(\Omega)$$

and

$$-\Delta_r v \geq \frac{1}{v^\delta} + g(x, v) \quad \text{weakly in } W^{-1, \frac{r}{r-1}}(\Omega).$$

Suppose in addition that there exist a positive function $u_0 \in L^\infty(\Omega)$ and positive constants c, C such that $cu_0 \leq u, v \leq Cu_0$ and

$$\int_{\Omega} |g(x, cu_0)|u_0 \, dx < +\infty, \quad \int_{\Omega} |g(x, Cu_0)|u_0 \, dx < +\infty. \tag{3.1}$$

Then, $u \leq v$.

Proof. First, for $\epsilon > 0$, we set $u_\epsilon \stackrel{\text{def}}{=} u + \epsilon$ and $v_\epsilon \stackrel{\text{def}}{=} v + \epsilon$. Following some ideas in Lindqvist [28] (see also Drábek and Hernández [17]) we use the Díaz–Saá inequality (see Díaz and Saá [15]) in the following way:

Let

$$\phi \stackrel{\text{def}}{=} \frac{u_\epsilon^r - v_\epsilon^r}{u_\epsilon^{r-1}} \quad \text{and} \quad \psi \stackrel{\text{def}}{=} \frac{v_\epsilon^r - u_\epsilon^r}{v_\epsilon^{r-1}}.$$

Setting $\Omega^+ = \{x \in \Omega \mid u(x) > v(x)\}$, we have that $\phi \geq 0$ and $\psi \leq 0$ in Ω^+ and

$$\begin{aligned} \int_{\Omega^+} |\nabla u|^{r-2} \nabla u \nabla \phi \, dx &\leq \int_{\Omega^+} \frac{\phi}{u^\delta} \, dx + \int_{\Omega^+} g(x, u) \phi \, dx < +\infty, \\ \int_{\Omega^+} |\nabla v|^{r-2} \nabla v \nabla \psi \, dx &\leq \int_{\Omega^+} \frac{\psi}{v^\delta} \, dx + \int_{\Omega^+} g(x, v) \psi \, dx < +\infty. \end{aligned}$$

Since

$$\nabla \phi = \left[1 + (r-1) \left(\frac{u_\epsilon}{v_\epsilon} \right)^r \right] \nabla u - r \left(\frac{v_\epsilon}{u_\epsilon} \right)^{r-1} \nabla v$$

and

$$\nabla \psi = \left[1 + (r-1) \left(\frac{v_\epsilon}{u_\epsilon} \right)^r \right] \nabla v - r \left(\frac{u_\epsilon}{v_\epsilon} \right)^{r-1} \nabla u,$$

we have

$$\begin{aligned} &\int_{\Omega^+} |\nabla u|^{r-2} \nabla u \nabla \phi \, dx + \int_{\Omega^+} |\nabla v|^{r-2} \nabla v \nabla \psi \, dx \\ &= \int_{\Omega^+} \left(|\nabla u_\epsilon|^r \left[1 + (r-1) \left(\frac{v_\epsilon}{u_\epsilon} \right)^r \right] + |\nabla v_\epsilon|^r \left[1 + (r-1) \left(\frac{u_\epsilon}{v_\epsilon} \right)^r \right] \right) dx \\ &= \int_{\Omega^+} (u_\epsilon^r - v_\epsilon^r) (|\nabla \log u_\epsilon|^r - |\nabla \log v_\epsilon|^r) \, dx \end{aligned}$$

$$\begin{aligned}
 & - \int_{\Omega^+} r v_\epsilon^r |\nabla \log u_\epsilon|^{r-2} \nabla \log u_\epsilon (\nabla \log v_\epsilon - \nabla \log u_\epsilon) \, dx \\
 & - \int_{\Omega^+} r u_\epsilon^r |\nabla \log v_\epsilon|^{r-2} \nabla \log v_\epsilon (\nabla \log u_\epsilon - \nabla \log v_\epsilon) \, dx.
 \end{aligned}$$

If $r \geq 2$, then using the following well-known inequality

$$|w_2|^r \geq |w_1|^r + r|w_1|^{r-2} w_1 (w_2 - w_1) + \frac{|w_2 - w_1|^r}{2^{r-1} - 1},$$

for all points w_1 and $w_2 \in \mathbb{R}^N$, we get that

$$\begin{aligned}
 & \int_{\Omega^+} |\nabla u|^{r-2} \nabla u \nabla \phi \, dx + \int_{\Omega^+} |\nabla v|^{r-2} \nabla v \nabla \psi \, dx \\
 & \geq \int_{\Omega^+} (u_\epsilon^r - v_\epsilon^r) (|\nabla \log u_\epsilon|^r - |\nabla \log v_\epsilon|^r) \, dx \\
 & \quad + \int_{\Omega^+} \left(v_\epsilon^r (|\nabla \log u_\epsilon|^r - |\nabla \log v_\epsilon|^r) + \frac{|\nabla \log v_\epsilon - \nabla \log u_\epsilon|^r}{2^{r-1} - 1} v_\epsilon^r \right) \, dx \\
 & \quad + \int_{\Omega^+} \left(u_\epsilon^r (|\nabla \log v_\epsilon|^r - |\nabla \log u_\epsilon|^r) + \frac{|\nabla \log u_\epsilon - \nabla \log v_\epsilon|^r}{2^{r-1} - 1} u_\epsilon^r \right) \, dx \\
 & \geq \frac{1}{2^{r-1} - 1} \int_{\Omega^+} |\nabla \log v_\epsilon - \nabla \log u_\epsilon|^r (v_\epsilon^r + u_\epsilon^r) \, dx \\
 & = \frac{1}{2^{r-1} - 1} \int_{\Omega^+} |u_\epsilon \nabla v_\epsilon - v_\epsilon \nabla u_\epsilon|^r \left(\frac{1}{u_\epsilon^r} + \frac{1}{v_\epsilon^r} \right) \, dx.
 \end{aligned}$$

If $1 < r < 2$ then using the following inequality (with some suitable $C(r) > 0$)

$$|w_2|^r \geq |w_1|^r + r|w_1|^{r-2} w_1 (w_2 - w_1) + C(r) \frac{|w_2 - w_1|^2}{(|w_1| + |w_2|)^{2-r}},$$

for all points w_1 and $w_2 \in \mathbb{R}^N$, the last term of the above inequality dealing with $r \geq 2$ is replaced by the following term

$$\begin{aligned}
 & C(r) \int_{\Omega^+} \frac{|\nabla \log v_\epsilon - \nabla \log u_\epsilon|^2}{(|\nabla \log v_\epsilon| + |\nabla \log u_\epsilon|)^{2-r}} (v_\epsilon^r + u_\epsilon^r) \, dx \\
 & = C(r) \int_{\Omega^+} \left(\frac{1}{u_\epsilon^r} + \frac{1}{v_\epsilon^r} \right) \frac{|u_\epsilon \nabla v_\epsilon - v_\epsilon \nabla u_\epsilon|^2}{(u_\epsilon |\nabla v_\epsilon| + v_\epsilon |\nabla u_\epsilon|)^{2-r}} \, dx.
 \end{aligned}$$

In the right-hand side, we get

$$\begin{aligned} & \int_{\Omega^+} \left(\frac{1}{u^\delta} + g(x, u) \right) \phi \, dx + \int_{\Omega^+} \left(\frac{1}{v^\delta} + g(x, v) \right) \psi \, dx \\ & \leq \int_{\Omega^+} \left[\frac{g(x, u)}{u^{r-1}} \left(\frac{u}{u_\epsilon} \right)^{r-1} - \frac{g(x, v)}{v^{r-1}} \left(\frac{v}{v_\epsilon} \right)^{r-1} \right] (u_\epsilon^r - v_\epsilon^r) \, dx. \end{aligned}$$

Then, since $\frac{u}{u_\epsilon} \rightarrow 1, \frac{v}{v_\epsilon} \rightarrow 1$ as $\epsilon \rightarrow 0^+$ a.e. in Ω , we get from (3.1) and the Lebesgue Theorem

$$\lim_{\epsilon \rightarrow 0^+} \int_{\Omega^+} (g(x, u)\phi + g(x, v)\psi) \, dx \leq 0.$$

By Fatou Lemma and using the above estimates, we obtain that $|u\nabla v - v\nabla u| = 0$ a.e. in Ω^+ from which we get that on each connected component set \mathcal{O} of Ω^+ , there exists $k \in \mathbb{R}$ such that $u = kv$ in \mathcal{O} . From

$$\begin{aligned} \int_{\mathcal{O}} |\nabla u|^r \, dx &= k^r \int_{\mathcal{O}} |\nabla v|^r \, dx \leq \int_{\mathcal{O}} (k^{1-\delta} v^{1-\delta} + g(x, kv)kv) \, dx, \\ k^r \int_{\mathcal{O}} |\nabla v|^r \, dx &\geq k^r \int_{\mathcal{O}} (v^{1-\delta} + g(x, v)v) \, dx \geq \int_{\mathcal{O}} (k^r v^{1-\delta} + g(x, kv)kv) \, dx \end{aligned}$$

we get $k \leq 1$ which implies that $u \leq v$ in Ω^+ and from the definition of $\Omega^+, u \leq v$ in Ω . \square

We now prove Theorem 0.8.

Proof of Theorem 0.8. For $0 < \alpha_f < \ell < \lambda_1(\Omega), A > 0$ large enough and $0 < \eta < M$ let define

$$\phi = \begin{cases} V & \text{if } \delta < 1, \\ \phi_1 (\ln(\frac{A}{\phi_1}))^{\frac{1}{p}} & \text{if } \delta = 1, \\ \phi_1^{\frac{p}{p-1+\delta}} & \text{if } \delta > 1, \end{cases} \quad \text{and } \underline{u} = \eta\phi, \tag{3.2}$$

with V a positive solution of

$$\begin{cases} -\Delta_p u = \ell u^{p-1} + \frac{1}{u^\delta} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The existence of V follows from similar minimization and cut-off arguments given in Giacomoni, Schindler and Takáč [21, proof of Lem. 3.3, p. 126]. Note that from $\ell < \lambda_1(\Omega)$, the associated energy functional is coercive and weakly lower semicontinuous in $W_0^{1,p}(\Omega)$ and from Lemma A.2 in [21] Gâteaux-differentiable in $W_0^{1,p}(\Omega)$. From Lemma A.6 and Theorem B.1 in [21] and since $\delta < 1, V \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$.

$$\bar{u} = \begin{cases} MV & \text{if } \delta < 1, \\ M\phi_1 (\ln(\frac{A}{\phi_1}))^{\frac{1}{p}} & \text{if } \delta = 1, \\ M(\phi_1 + \phi) & \text{if } \delta > 1. \end{cases} \tag{3.3}$$

We verify that $\underline{u} \in \mathcal{C}, \bar{u} \in \mathcal{C}$. Let $L > 0$ such that $-L \leq f(x, s) \leq \ell s^{p-1} + L$. We verify that for $M > 0$ large enough for $\eta > 0$ small enough we have

$$-\Delta_p \underline{u} - \frac{1}{\underline{u}^\delta} \leq -L \quad \text{in } \Omega, \quad \underline{u} = 0 \quad \text{on } \partial\Omega,$$

and

$$-\Delta_p \bar{u} - \frac{1}{\bar{u}^\delta} \geq \ell \bar{u}^{p-1} + L \quad \text{in } \Omega, \quad \bar{u} = 0 \quad \text{on } \partial\Omega.$$

We distinguish between the following two cases: the case where $\delta < 1$ and the case where $\delta \geq 1$. In the first case, the solution $u \in W_0^{1,p}(\Omega)$ to (Q) can be obtained as a global minimizer in $W_0^{1,p}(\Omega)$ of the functional E defined below at $v \in W_0^{1,p}(\Omega)$:

$$E(v) \stackrel{\text{def}}{=} \frac{1}{p} \int_{\Omega} |\nabla v|^p \, dx - \int_{\Omega} G(x, v) \, dx - \int_{\Omega} K(x, v) \, dx,$$

where for any $t \geq 0$

$$G(x, t) \stackrel{\text{def}}{=} \int_0^t g(x, s) \, ds, \quad K(x, t) \stackrel{\text{def}}{=} \int_0^t k(x, s) \, ds,$$

and g, k are the cut-off functions defined by

$$g(x, v(x)) \stackrel{\text{def}}{=} \begin{cases} v(x)^{-\delta} & \text{if } v(x) \geq \underline{u}(x), \\ \underline{u}(x)^{-\delta} & \text{otherwise,} \end{cases}$$

and

$$k(x, v(x)) \stackrel{\text{def}}{=} \begin{cases} f(x, \underline{u}(x)) & \text{if } v(x) \leq \underline{u}(x), \\ f(x, v(x)) & \text{if } \underline{u} \leq v(x) \leq \bar{u}(x), \\ f(x, \bar{u}(x)) & \text{if } v(x) \geq \bar{u}(x). \end{cases}$$

Notice that the method of proof of Theorem 0.5 when $\delta < 1$ does not apply here because $\int_{\Omega} F(x, v) \, dx$ with $F(x, t) = \int_0^t f(x, s) \, ds$ is not convex in v . That is the reason why we introduce the above cut-off function. Since f satisfies (0.1), E is coercive and weakly lower semicontinuous in $W_0^{1,p}(\Omega)$. Using the compactness of any minimizing sequence $\{u_n\}$ in $L^p(\Omega)$ and the Lebesgue Theorem, we can prove the existence of a global minimizer u to E . From Lemma A.2 in [21], we have that E is Gâteaux-differentiable in u and then u satisfies

$$\begin{cases} -\Delta_p u - g(x, u) = k(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial. \end{cases}$$

Thus, from the weak comparison principle, we first get that $\underline{u} \leq u$ and then that $g(x, u) = u^{-\delta}$. Finally, still from the weak comparison principle we also obtain $u \leq \bar{u}$ from which we get $k(x, u) = f(x, u)$ and $u \in \mathcal{C}$.

Now, we deal with the second case. We use the following iterative scheme

$$\begin{aligned}
 -\Delta_p u_n - \frac{1}{u_n^\delta} + Ku_n &= f(x, u_{n-1}) + Ku_{n-1} \quad \text{in } \Omega, \\
 u_n &= 0 \quad \text{on } \partial\Omega,
 \end{aligned}$$

with $u_0 \stackrel{\text{def}}{=} \underline{u}$ and $K > 0$ large enough such that $t \rightarrow Kt + f(x, t)$ is nondecreasing (thanks to the uniform local Lipschitz property of f) in $[0, \|\bar{u}\|_{L^\infty(\Omega)}]$ for a.e. $x \in \Omega$. Note that the iterative scheme is well defined and produces a sequence of element $u_n \in W_0^{1,p}(\Omega) \cap \mathcal{C} \cap C_0(\bar{\Omega})$. From the weak comparison principle, we have that $(u_n)_{n \in \mathbb{N}}$ is a monotone increasing sequence such that $u_n \leq \bar{u}$. Then $u_n \uparrow u$ in $C_0(\bar{\Omega}) \cap \mathcal{C}$ and using the equation satisfied by u_n we deduce that $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $W_0^{1,p}(\Omega)$ and then converges to u in $W_0^{1,p}(\Omega)$. Thus, by passing to the limit in the equation satisfied by u_n we obtain that u is a solution to (Q). Finally, the uniqueness of u follows from Theorem 3.1. \square

We now give the proof of Theorem 0.13.

Proof of Theorem 0.13. Let $T > 0$, $N \in \mathbb{N}^*$ and $\Delta_t \stackrel{\text{def}}{=} \frac{T}{N}$. Following the main steps of the previous section, we are interested in constructing the sequence $(u^n)_{n \in \mathbb{N}} \subset L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$, solutions to

$$u^n - \Delta_t \left(\Delta_p u^n + \frac{1}{(u^n)^\delta} \right) = \Delta_t f(x, u^{n-1}) + u^{n-1} \quad \text{in } \Omega. \tag{3.4}$$

Applying Theorem 0.5 for each iteration n and since $u_0 \in \mathcal{C} \cap W_0^{1,p}(\Omega)$, we get the existence of $(u_n)_{n \in \mathbb{N}} \subset \mathcal{C} \cap W_0^{1,p}(\Omega)$. In fact, the previous inclusion is uniform in Δ_t . Indeed, since $u_0 \in \mathcal{C}$ we can choose \underline{u} and \bar{u} defined by (3.2) and (3.3) with $\eta > 0$ and $M > 0$ large enough so that $\underline{u} \leq u_0 \leq \bar{u}$. Then with $f \geq -L$ the weak comparison principle guarantees $\underline{u} \leq u_n \leq \bar{u}$, with \underline{u} and \bar{u} independent on Δ_t .

Next, let u_{Δ_t} and \tilde{u}_{Δ_t} defined by (2.4) and set $u_{\Delta_t}(t) = u_0$ if $t < 0$. Then (2.5) is satisfied with $h_{\Delta_t}(t) \stackrel{\text{def}}{=} f(x, u_{\Delta_t}(t - \Delta_t))$ on $[0, T]$. Notice that from $u_{\Delta_t} \in [\underline{u}, \bar{u}]$ and $t \rightarrow f(x, t)$ continuous on $[\underline{u}, \bar{u}]$ it follows that $h_{\Delta_t}(t)$ is bounded in $L^\infty(Q_T)$ independently on n . Then by similar energy estimates as in the proof of Theorem 0.9 we get that

$$\begin{aligned}
 u_{\Delta_t}, \tilde{u}_{\Delta_t} &\in L^\infty(0, T; W_0^{1,p}(\Omega) \cap \mathcal{C}), \\
 \frac{\partial \tilde{u}_{\Delta_t}}{\partial t} &\in L^2(Q_T), \\
 u_{\Delta_t}, \tilde{u}_{\Delta_t} &\in L^\infty(Q_T), \\
 \frac{1}{(u_{\Delta_t})^\delta} &\in L^\infty(0, T; W^{-1,p'}(\Omega)), \\
 \|\tilde{u}_{\Delta_t} - u_{\Delta_t}\|_{L^2(\Omega)} &\leq C(\Delta_t)^{\frac{1}{2}},
 \end{aligned} \tag{3.5}$$

uniformly on Δ_t . Then, taking $\Delta_t \rightarrow 0$, it follows that there exists $u \in L^\infty(0, T; W_0^{1,p}(\Omega))$ such that $u \in L^\infty(Q_T)$ and, up to a subsequence, we have

$$\begin{aligned}
 u_{\Delta_t}, \tilde{u}_{\Delta_t} &\xrightarrow{*} u \quad \text{in } L^\infty(0, T; W_0^{1,p}(\Omega)) \text{ and in } L^\infty(Q_T), \\
 \frac{\partial \tilde{u}_{\Delta_t}}{\partial t} &\rightharpoonup \frac{\partial u}{\partial t} \quad \text{in } L^2(Q_T).
 \end{aligned} \tag{3.6}$$

Using similar arguments as in the proof of Theorem 0.9, we get for $1 < q < +\infty$,

$$u_{\Delta t}, \tilde{u}_{\Delta t} \rightarrow u \text{ in } L^\infty(0, T; L^q(\Omega)) \text{ and } u \in C([0, T], L^q(\Omega)). \tag{3.7}$$

Moreover, if $K > 0$ denotes the Lipschitz constant of f on $[\underline{u}, \bar{u}]$ we have

$$\|f(x, u_{\Delta t}(t - \Delta t)) - f(x, u(t))\|_{L^2(\Omega)} \leq K \|u_{\Delta t}(t - \Delta t) - u(t)\|_{L^2(\Omega)},$$

and from (3.7) we deduce that $h_{\Delta t} = f(x, u_{\Delta t}(\cdot - \Delta t)) \rightarrow f(x, u)$ in $L^\infty(0, T; L^2(\Omega))$. Then by following the steps at the end of the proof of Theorem 0.9 we obtain that u is a weak solution to (P_t) in $\mathbf{V}(Q_T)$.

Next, let us prove that such a solution is unique. For that, let v be a weak solution to (P_t) in $\mathbf{V}(Q_T)$. Since $f(x, \cdot)$ is locally Lipschitz uniformly in Ω , it follows that

$$\begin{aligned} & \frac{1}{2} \|u - v\|_{L^2(\Omega)}^2 - \int_0^T \langle \Delta_p u - \Delta_p v, u - v \rangle dt - \int_0^T \int_\Omega \left(\frac{1}{u^\delta} - \frac{1}{v^\delta} \right) (u - v) dx dt \\ &= \int_0^T \int_\Omega (f(x, u) - f(x, v)) dx dt \\ &\leq C \int_0^T \int_\Omega |u - v|^2 dx dt \end{aligned}$$

which implies together with the Gronwall Lemma and (1.3), that $u \equiv v$. Finally, as in the proof of Theorem 0.9, we can prove that $u \in C([0, T]; W_0^{1,p}(\Omega))$ and that u satisfies (0.7). \square

We now give the proof of Proposition 0.2.

Proof of Proposition 0.2. (i) is the consequence of (0.5) together with the fact that f is locally Lipschitz and the Gronwall Lemma.

Regarding assertion (ii), we follow the proof of Proposition 0.1: Assume without loss of generality that $t > s$. Then,

$$\|u(t) - u(s)\|_{L^\infty(\Omega)} \leq \|u_0 - u(t - s)\|_{L^\infty(\Omega)} + \int_0^s \|f(x, u(\tau)) - f(x, u(\tau + t - s))\|_{L^\infty(\Omega)} d\tau.$$

From assertion (i) and the fact that f is Lipschitz on $[\underline{u}, \bar{u}]$, it follows that

$$\begin{aligned} \|u(t) - u(s)\|_{L^\infty(\Omega)} &\leq \|u_0 - u(t - s)\|_{L^\infty(\Omega)} + \omega \int_0^s e^{\omega\tau} d\tau \|u_0 - u(t - s)\|_{L^\infty(\Omega)} \\ &\leq e^{\omega s} \|u_0 - u(t - s)\|_{L^\infty(\Omega)}. \end{aligned}$$

Now, we estimate the term $\|u_0 - u(t - s)\|_{L^\infty(\Omega)}$ in the following way:

$$\begin{aligned} \|u_0 - u(t-s)\|_{L^\infty(\Omega)} &\leq \int_0^{t-s} \|Au_0 - f(x, u(\tau))\|_{L^\infty(\Omega)} \, d\tau \\ &\leq (t-s) \|Au_0 - f(x, u_0)\|_{L^\infty(\Omega)} + \omega \int_0^{t-s} \|u_0 - u(\tau)\|_{L^\infty(\Omega)} \, d\tau. \end{aligned}$$

From Gronwall Lemma, we deduce that

$$\|u_0 - u(t-s)\|_{L^\infty(\Omega)} \leq (t-s)e^{\omega(t-s)} \|Au_0 - f(x, u_0)\|_{L^\infty(\Omega)}$$

Gathering the above estimates, we get

$$\|u(t) - u(s)\|_{L^\infty(\Omega)} \leq (t-s)e^{\omega t} \|Au_0 - f(x, u_0)\|_{L^\infty(\Omega)}.$$

Then, the rest of the proof follows with the same arguments as in the proof of Proposition 0.1. \square

To end this section, we prove Theorem 0.15.

Proof of Theorem 0.15. Let $\underline{u}, \bar{u} \in C \cap W_0^{1,p}(\Omega) \cap C(\bar{\Omega})$ be the subsolution and supersolution to

$$\begin{cases} -\Delta_p u - \frac{1}{u^\delta} = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

which are defined by (3.2) and (3.3) where $\eta > 0$ is small enough and $M > 0$ is large enough so that $\underline{u} \leq u_0 \leq \bar{u}$. Note that it is possible since $u_0 \in C \cap W_0^{1,p}(\Omega)$. Thus, let u be the solutions to (P_t) and u_1, u_2 the solutions to (P_t) with initial data $u_0 = \underline{u}$ and $u_0 = \bar{u}$ respectively, see Theorem 0.13. From (3.2) and (3.3), we have that

$$\underline{u}, \bar{u} \in \overline{\mathcal{D}(A)}^{L^\infty(\Omega)}. \tag{3.9}$$

Indeed, let $f, g \in W^{-1, \frac{p}{p-1}}(\Omega)$ defined by $f \stackrel{\text{def}}{=} A\underline{u} \leq 0, g \stackrel{\text{def}}{=} A\bar{u} \geq 0$ and $(u_n)_{n \in \mathbb{N}}, (v_n)_{n \in \mathbb{N}}$ two sequences of $\mathcal{D}(A)$ defined by

$$Au_n = f_n \stackrel{\text{def}}{=} \max\{f, -n\}, \quad Av_n = g_n \stackrel{\text{def}}{=} \min\{g, n\}.$$

From the weak comparison principle, we have that $(u_n)_{n \in \mathbb{N}}$ is nonincreasing and $(v_n)_{n \in \mathbb{N}}$ is nondecreasing. Moreover, since $f_n \rightarrow f$ and $g_n \rightarrow g$ in $W^{-1, \frac{p}{p-1}}(\Omega)$ as $n \rightarrow +\infty, u_n \rightarrow \underline{u}$ and $v_n \rightarrow \bar{u}$ in $W_0^{1,p}(\Omega)$ as $n \rightarrow +\infty$. Therefore, $u_n \rightarrow \underline{u}$ and $v_n \rightarrow \bar{u}$ a.e. in Ω as $n \rightarrow \infty$. Consequently, using Dini's Theorem, we get that $u_n \rightarrow \underline{u}$ and $v_n \rightarrow \bar{u}$ in $L^\infty(\Omega)$ as $n \rightarrow +\infty$. From (3.9) and Theorem 0.13, we obtain that $u_1(t)$ and $u_2(t) \in C([0, T]; C_0(\bar{\Omega}))$. Furthermore, since $\underline{u}, \bar{u} \in C$ are subsolution and supersolution respectively to (3.8), we have that the sequence $(\underline{u}^n)_{n \in \mathbb{N}}$ (resp. $(\bar{u}^n)_{n \in \mathbb{N}}$) defined in (3.4) with $u_0 = \underline{u}$ ($u_0 = \bar{u}$ resp.) is nondecreasing (nonincreasing resp.) for any $0 < \Delta_t < 1/K$ where $K > 0$ is the Lipschitz constant of f on $[\underline{u}, \bar{u}]$. Moreover, the sequence $(u^n)_{n \in \mathbb{N}}$ defined by (3.4) satisfies $\underline{u}^n \leq u^n \leq \bar{u}^n$ and it follows that $u_1(t) \leq u(t) \leq u_2(t)$ and that $t \rightarrow u_1(t)$ ($t \rightarrow u_2(t)$ resp.) is nondecreasing (nonincreasing resp.) and converges a.e. in Ω to u_1^∞ (u_2^∞ resp.), as $t \rightarrow \infty$. From the semigroup theory we have $u_1^\infty = \lim_{t' \rightarrow +\infty} S(t' + t)(\underline{u}) = S(t)(\lim_{t' \rightarrow +\infty} S(t')\underline{u}) = S(t)u_1^\infty$ and analogously $u_2^\infty = S(t)u_2^\infty$, where $S(t)$ is the semigroup on $L^\infty(\Omega)$ generated by the evolution equation, and then

u_1^∞ and u_2^∞ are stationary solutions to (P_t) . From Theorem 0.8, we get that $u_1^\infty = u_2^\infty = u_\infty \in C(\overline{\Omega})$. Therefore, from Dini’s Theorem we get that

$$u_1(t) \rightarrow u_\infty, u_2(t) \rightarrow u_\infty \quad \text{in } L^\infty(\Omega) \text{ as } t \rightarrow \infty, \tag{3.10}$$

and then (0.8) follows since $u_1(t) \leq u(t) \leq u_2(t)$. \square

4. The non-degenerate case: $p = 2$

We start by proving the first part, as well as points (iv) and (v), of Theorem 0.16, namely:

Theorem 4.1. *Let $0 < \delta, T > 0, u_0 \in \mathcal{C}$ and let f satisfy assumptions in Theorem 0.13. Then there exists a unique solution $u \in C([0, T]; L^2(\Omega)) \cap L^\infty(Q_T)$ such that $u(t) \in \mathcal{C}$ a.e. $t \in [0, T]$, in the sense of distributions to*

$$\begin{cases} u_t - \Delta u - \frac{1}{u^\delta} = f(x, u) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \quad u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{4.1}$$

Moreover, points (iv) and (v) of Theorem 0.16 are satisfied.

Proof. First, for $g \in L^\infty(Q_T)$ let us prove the existence of a weak solution $u \in C([0, T]; L^2(\Omega)) \cap L^\infty(Q_T)$ in the sense of distributions to

$$\begin{cases} u_t - \Delta u - \frac{1}{u^\delta} = g & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \quad u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{4.2}$$

Since $u_0 \in \mathcal{C}$, for any $\epsilon > 0$ there exists $\underline{u}_\epsilon \in C_0(\overline{\Omega}) \cap H_0^1(\Omega)$ in the form (1.5) and (1.7) with $p = 2$ if $\delta \geq 1$ and $\underline{u}_\epsilon = \eta\phi_1$ if $\delta < 1$ (with $\eta > 0$ small enough) such that $\underline{u}_\epsilon \leq u_0$ and

$$-\Delta \underline{u}_\epsilon - \frac{1}{(\underline{u}_\epsilon + \epsilon)^\delta} \leq -\|g\|_{L^\infty(Q_T)} \quad \text{in } \Omega.$$

In addition, there exists \bar{u} in the form given by (3.3) such that $u_0 \leq \bar{u}$ and verifying

$$-\Delta \bar{u} - \frac{1}{\bar{u}^\delta} \geq \|g\|_{L^\infty(Q_T)} \quad \text{in } \Omega.$$

Then following the method and using estimates from Section 1 (in particular (2.6) with $p = 2$), we can prove the existence and the uniqueness of a positive solution $u_\epsilon \in L^\infty(Q_T) \cap L^2(0, T; H_0^1(\Omega))$, satisfying $\underline{u}_\epsilon \leq u_\epsilon \leq \bar{u}$, to

$$(P_{\epsilon,t}) \quad \begin{cases} u_t - \Delta u - \frac{1}{(u + \epsilon)^\delta} = g & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \quad u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{4.3}$$

From the weak comparison principle, we get that if $0 \leq \tilde{\epsilon} \leq \epsilon$ then $u_\epsilon \leq u_{\tilde{\epsilon}}$ and $u_{\tilde{\epsilon}} + \tilde{\epsilon} \leq u_\epsilon + \epsilon$. This last inequality is obtained by remarking that $v_\epsilon \stackrel{\text{def}}{=} u_\epsilon + \epsilon$ and $v_{\tilde{\epsilon}} \stackrel{\text{def}}{=} u_{\tilde{\epsilon}} + \tilde{\epsilon}$ obeys

$$\begin{cases} (v_{\bar{\epsilon}} - v_{\epsilon})_t - \Delta(v_{\bar{\epsilon}} - v_{\epsilon}) - \frac{1}{(v_{\bar{\epsilon}})^{\delta}} + \frac{1}{(v_{\epsilon})^{\delta}} = 0 & \text{in } Q_T, \\ v_{\bar{\epsilon}} - v_{\epsilon} \leq 0 & \text{on } \Sigma_T, \quad \text{and} \quad v_{\bar{\epsilon}}(0) - v_{\epsilon}(0) \leq 0 & \text{in } \Omega. \end{cases}$$

It follows that $(u_{\epsilon})_{\epsilon>0}$ is a Cauchy sequence in $L^{\infty}(Q_T)$ and there exists $u \in L^{\infty}(Q_T)$ satisfying $\underline{u} \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow 0^+} u_{\epsilon} \leq u \leq \bar{u}$ and such that $u_{\epsilon} \rightarrow u$ in $L^{\infty}(Q_T)$ as $\epsilon \rightarrow 0^+$. Therefore, u is uniformly in C . Then, passing to the limit as $\epsilon \rightarrow 0^+$, it is easy to get from (4.3) that u is a solution in the sense of distributions to (4.2). Finally, since $(u_{\epsilon})_{\epsilon>0}$ is also a Cauchy sequence in $L^{\infty}(0, T; L^2(\Omega))$, and since one has $u_{\epsilon} \in C([0, T]; L^2(\Omega))$ (by regularity results for the heat equation) then $u \in C([0, T]; L^2(\Omega))$. To prove the uniqueness of u , let us suppose that $v \in L^{\infty}(Q_T) \cap C([0, T], L^2(\Omega))$, $v(t) \in C$ a.e., is another solution (in the sense of distributions) to (4.2). Then subtracting the equation satisfied by v to (4.2) and doing the dual product in $H^2(\Omega) \cap H_0^1(\Omega)$ with $(-\Delta)^{-1}(u - v)$ yield

$$\begin{aligned} & \frac{d}{dt} \|u(t) - v(t)\|_{H^{-1}(\Omega)}^2 + 2\|u(t) - v(t)\|_{L^2(\Omega)}^2 \\ & + 2 \int_{\Omega} \left(\frac{1}{v(t)^{\delta}} - \frac{1}{u(t)^{\delta}} \right) (-\Delta)^{-1}(u(t) - v(t)) \, dx = 0, \end{aligned}$$

where $\|\cdot\|_{H^{-1}(\Omega)} \stackrel{\text{def}}{=} \sqrt{\langle (-\Delta)^{-1} \cdot, \cdot \rangle_{H_0^1(\Omega), H^{-1}(\Omega)}}$ is a norm on $H^{-1}(\Omega)$. Note that from Hardy's Inequality the last above term is well defined since for a.e. $t \in [0, T]$ we have $(-\Delta)^{-1}(u(t) - v(t)) \in H^2(\Omega) \cap H_0^1(\Omega)$ $u(t), v(t) \in C$ implies $\frac{1}{v(t)^{\delta}} - \frac{1}{u(t)^{\delta}} \in H^{-\frac{3}{2}+\eta}(\Omega) \subset (H^2(\Omega) \cap H_0^1(\Omega))'$ for $0 < \eta < \frac{2}{1+\delta}$ because $H^2(\Omega) \cap H_0^1(\Omega)$ is densely and continuously (compactly) embedded into $H_0^{\frac{3}{2}-\eta}(\Omega)$ (see Grisvard [23] and Lemma (4.5) below), and it is positive since $(-\Delta)^{-1}$ is monotone. Then integrating over $(0, T)$ and taking into account $u(0) = v(0) = u_0 \in H^{-1}(\Omega)$ yields $u \equiv v$.

Let us now prove the existence and uniqueness of the solution to (4.1). For that we now consider \underline{u}, \bar{u} defined by (3.2), (3.3) (subsolution and supersolution to the stationary equation (4.2), resp.) and obeying $\underline{u} \leq u_0 \leq \bar{u}$. For z_1, z_2 in $\{\underline{u} \leq u \leq \bar{u}\} \cap C([0, T]; L^2(\Omega))$ we consider u_{ϵ}^1 (resp. u_{ϵ}^2) the solution to $(P_{\epsilon,t})$ for $g = f(x, z_1)$ ($g = f(x, z_2)$, resp.). By multiplying by $u_{\epsilon}^1 - u_{\epsilon}^2$ the equation satisfied by $u_{\epsilon}^1 - u_{\epsilon}^2$, integrating over $(0, t)$, and taking into account that $f(x, \cdot)$ is Lipschitz in $[0, \|\bar{u}\|_{\infty}]$, we obtain

$$\int_{\Omega} |u_{\epsilon}^1(t) - u_{\epsilon}^2(t)|^2 \, dx + 2 \int_0^t \int_{\Omega} |\nabla(u_{\epsilon}^1 - u_{\epsilon}^2)|^2 \, dx \, dt \leq 2\omega \int_0^t \int_{\Omega} |z_1(t) - z_2(t)|^2 \, dx \, dt,$$

for a positive constant $\omega > 0$ depending on $\|\bar{u}\|_{\infty}$ but not in z_1, z_2 . From the weak comparison principle we deduce that $\underline{u}_{\epsilon} \leq u_{\epsilon}^i \leq \bar{u}$, $i = 1, 2$, and by passing to the limit $\epsilon \rightarrow 0^+$ we obtain that the solution to (4.2) for $g = f(x, z_1)$ (resp. $g = f(x, z_2)$) obeys:

$$\int_{\Omega} |u^1(t) - u^2(t)|^2 \, dx \leq 2\omega \int_0^t \int_{\Omega} |z_1(t) - z_2(t)|^2 \, dx \, dt \quad \text{and} \quad \underline{u} \leq u^i \leq \bar{u}, \quad i = 1, 2. \quad (4.4)$$

Then by applying the fixed point theorem in $\{\underline{u} \leq u \leq \bar{u}\} \cap C([0, T]; L^2(\Omega))$ for $T = T(\|\bar{u}\|_{\infty}) > 0$ small enough, we get the existence of a weak solution (in the sense of distributions), $u \in \{\underline{u} \leq u \leq \bar{u}\} \cap C([0, T]; L^2(\Omega))$, to (0.9). Thus, since $f(x, \cdot)$ is Lipschitz in $[0, \|\bar{u}\|_{\infty}]$, we can extend u in $[0, +\infty)$ and $u \in C([0, +\infty); L^2(\Omega))$. Note that the uniqueness of the solution is an obvious consequence of (4.4). Finally, if u_{ϵ}^i is the solution to (4.3) with $u_0 = u_{0,i}$ and $g = f(x, u_i)$, $i = 1, 2$, then by multiplying by $u_{\epsilon}^1 - u_{\epsilon}^2$ the equation satisfied by $u_{\epsilon}^1 - u_{\epsilon}^2$ and integrating over $(0, t)$ we obtain

$$\int_{\Omega} |u_{\epsilon}^1(t) - u_{\epsilon}^2(t)|^2 dx + 2\lambda_1 \int_0^t \int_{\Omega} |u_{\epsilon}^1 - u_{\epsilon}^2|^2 dx dt$$

$$\leq 2 \int_0^t \int_{\Omega} (f(x, u_1) - f(x, u_2))(u_{\epsilon}^1 - u_{\epsilon}^2) dx dt + \int_{\Omega} (u_{0,1} - u_{0,2})^2 dx,$$

where $\lambda_1 > 0$ is the first eigenvalue of the Dirichlet Laplacian ($\int_{\Omega} |\nabla z|^2 dx \geq \lambda_1 \int_{\Omega} |z|^2 dx$ for all $z \in H_0^1(\Omega)$). Then (0.10) is obtained by passing to the limit $\epsilon \rightarrow 0^+$ and by applying Gronwall Lemma. \square

Next, we discuss the regularity of solutions to (4.2). Precisely, we prove the following result:

Theorem 4.2. *Under the assumptions of Theorem 4.1, $\forall \eta > 0$ small enough, we have that any solution u to (4.2) satisfies:*

- (i) *if $\delta < 1$ and $u_0 \in C \cap H^{2-\eta}(\Omega)$, then $u \in C([0, T]; H^{2-\eta}(\Omega))$;*
- (ii) *if $\frac{1}{2} \leq \delta < 1$ and $u_0 \in C \cap H^{\frac{5}{2}-\delta-\eta}(\Omega)$, then $u \in C([0, T]; H^{\frac{5}{2}-\delta-\eta}(\Omega))$;*
- (iii) *if $\delta \geq 1$ and $u_0 \in C \cap H^{\frac{1}{2}+\frac{2}{\delta+1}-\eta}(\Omega)$, then $u \in C([0, T]; H^{\frac{1}{2}+\frac{2}{\delta+1}-\eta}(\Omega))$.*

To prove Theorem 4.2, we will use the interpolation theory in Sobolev spaces (see Grisvard [23], Triebel [41]) and Hardy Inequalities. In this regard, we adopt the following notations: if X, Y are two Banach spaces, by $X \subset Y$ we mean that X is continuously imbedded in Y , for $\theta \in (0, 1)$, $1 \leq q \leq \infty$, we denote by $(X, Y)_{\theta, q}$ the interpolation space obtained from X and Y with the real method (see [41, p. 24]), and for $0 < T \leq \infty$, $1 \leq q \leq \infty$ we define the space

$$W_q(0, T, X, Y) \stackrel{\text{def}}{=} \{u \in L^q(0, T; X) \mid u_t \in L^q(0, T; Y)\}$$

equipped with the norm $(\int_0^T \|u\|_X^q + \|u_t\|_Y^q dt)^{\frac{1}{q}}$.

Moreover, let us recall some basic facts about fractional powers of $-\Delta$ with domain $\mathcal{D}(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$, i.e. $(-\Delta)^{\theta}$ with domain $\mathcal{D}((-\Delta)^{\theta})$ in $L^2(\Omega)$ for $\theta \in [0, 1]$ (see for instance [41, Par. 1.15.1, p. 98 and Par. 1.18.10, p. 141] or [33, Chap. 2, Par. 2.6] for the definition):

Proposition 4.1. *Let $\theta \in [0, 1]$.*

(1)
$$\mathcal{D}((-\Delta)^{\theta}) = (\mathcal{D}(\Delta), L^2(\Omega))_{1-\theta, 2}.$$

(2)
$$\mathcal{D}((-\Delta)^{\theta}) = \begin{cases} H^{2\theta}(\Omega) & \text{if } 0 \leq \theta < \frac{1}{4}, \\ \tilde{H}^{\frac{1}{2}}(\Omega) & \text{if } \theta = \frac{1}{4}, \\ H_0^{2\theta}(\Omega) & \text{if } \frac{1}{4} < \theta \leq 1, \end{cases}$$

where $\tilde{H}^{\frac{1}{2}}(\Omega) \stackrel{\text{def}}{=} \{v \in H^{\frac{1}{2}}(\Omega) \mid d(x)^{-\frac{1}{2}} v \in L^2(\Omega)\}$.

(3) $(-\Delta)^{\theta}$ is an isomorphism from $\mathcal{D}(-\Delta)$ onto $\mathcal{D}((-\Delta)^{1-\theta})$ as well as from $L^2(\Omega)$ onto $(\mathcal{D}((-\Delta)^{\theta}))'$ (the dual space of $(\mathcal{D}((-\Delta)^{\theta}))$).

Remark 4.3. Note that since we are in a Hilbertian framework, the real interpolation space $(\mathcal{D}(\Delta), L^2(\Omega))_{1-\theta, 2}$ coincides with the complex interpolation space $[\mathcal{D}(\Delta), L^2(\Omega)]_{1-\theta}$, see [41, Par. 1.9, p. 55 and Par. 1.8.10, Rem. 3 p. 143].

Proof. The first point follows from the fact that $-\Delta$ is positive and self-adjoint, see [41, Thm. 1.18.10.1, Par. 1.18.1, p. 141]. The second point is a consequence of the characterization of $(H^2(\Omega) \cap H_0^1(\Omega), L^2(\Omega))_{1-\theta, 2}$ obtained from [41, Thm. 4.3.3.1, Par. 4.3.3, p. 321] or from [22]. The first part of the last point follows from [41, Thm. 1.15.2.1, p. 101] and the second part is deduced with a duality argument (combined with the self-adjointness of $-\Delta$) from the fact that $-\Delta$ is an isomorphism from $\mathcal{D}((-\Delta)^\theta)$ onto $L^2(\Omega)$. \square

For $1 < q < \infty$, $0 < T \leq +\infty$ setting the Banach space

$$\mathcal{X}_{q,\theta,T} \stackrel{\text{def}}{=} W_q(0, T; \mathcal{D}((-\Delta)^{1-\theta}), (\mathcal{D}((-\Delta)^\theta))'),$$

we have the following result:

Lemma 4.4. Let $\theta \in [0, 1)$ and $q > \frac{2}{1-\theta}$. For $0 < T < +\infty$ let L_T be the linear operator defined by $L_T(f) \stackrel{\text{def}}{=} z$, where z is the solution to

$$\begin{cases} z_t - \Delta z = f & \text{in } Q_T, \\ z = 0 & \text{on } \Sigma_T; \quad z(0) = 0 & \text{in } \Omega. \end{cases}$$

Then L_T is a bounded operator from $L^q(0, T; (\mathcal{D}((-\Delta)^\theta))')$ into $\mathcal{X}_{q,\theta,T}$ as well as from $L^q(0, T; (\mathcal{D}((-\Delta)^\theta))')$ into $C([0, T], \mathcal{D}((-\Delta)^{1-\theta-\frac{2}{q}}))$.

Proof. Since Δ generates an analytic semigroup in $L^2(\Omega)$, then $L_{+\infty}$ is a continuous operator from $L^q(0, +\infty; L^2(\Omega))$ onto $W_q(0, +\infty; \mathcal{D}(-\Delta), L^2(\Omega))$ (see [13, Par. 4.2]). Therefore, since $(-\Delta)^\theta$ is an isomorphism from $\mathcal{D}(-\Delta)$ onto $\mathcal{D}((-\Delta)^{1-\theta})$ and from $L^2(\Omega)$ onto $\mathcal{D}((-\Delta)^\theta)'$ and from the fact that $(-\Delta)^\theta$ and $(\cdot)_t - \Delta$ commute, $L_{+\infty}$ is a continuous operator from $L^q(0, +\infty; L^2(\Omega))$ onto $\mathcal{X}_{q,\theta,+\infty}$. Then for $f \in L^q(0, T; L^2(\Omega))$, if we denote by \hat{f} the extension of f by zero for $t > T$, we have

$$\|L_T f\|_{\mathcal{X}_{q,\theta,T}} \leq \|L_\infty \hat{f}\|_{\mathcal{X}_{q,\theta,+\infty}} \leq C \|\hat{f}\|_{L^q(0,T;L^2(\Omega))} = C \|f\|_{L^q(0,T;L^2(\Omega))}$$

which gives the first statement of the lemma.

We prove now the second statement. The above inequalities with the continuous embedding of $\mathcal{X}_{q,\theta,+\infty}$ into $C([0, T], (\mathcal{D}((-\Delta)^{1-\theta}), (\mathcal{D}((-\Delta)^\theta))')_{\frac{1}{q},q})$ (see [41, Thm. 1.8.2.1(2), Par. 1.8.2, p. 44] or [7, Part II, Chap. 1, Par. 4.2]) guarantee:

$$\|L_T f\|_{C([0,T],(\mathcal{D}((-\Delta)^{1-\theta}),(\mathcal{D}((-\Delta)^\theta))')_{\frac{1}{q},q})} \leq C \|f\|_{L^q(0,T;L^2(\Omega))}.$$

Furthermore, from the interpolation theorem we have

$$(\mathcal{D}((-\Delta)^{1-\theta}), (\mathcal{D}((-\Delta)^\theta))')_{\frac{1}{q},q} = (-\Delta)^\theta ((\mathcal{D}(-\Delta), L^2(\Omega))_{\frac{1}{q},q}),$$

and from well-known properties of real interpolation spaces (see [41, Thm. 1.1.3.3, p. 25]) and from Propositions 4.1, we get

$$(\mathcal{D}(-\Delta), L^2(\Omega))_{\frac{1}{q},q} \subset (\mathcal{D}(-\Delta), L^2(\Omega))_{\frac{2}{q},1} \subset (\mathcal{D}(-\Delta), L^2(\Omega))_{\frac{2}{q},2} = \mathcal{D}((-\Delta)^{1-\frac{2}{q}}).$$

Then from above, it follows that $u \in C([0, T], \mathcal{D}((-\Delta)^{1-\theta-\frac{2}{q}}))$ and

$$\|L_T f\|_{C([0,T], \mathcal{D}((-\Delta)^{1-\theta-\frac{2}{q}}))} \leq C \|f\|_{L^q(0,T; L^2(\Omega))}. \quad \square$$

Let us also recall the following Hardy type inequalities which can be obtained from [41, Par. 3.2.6, Lem. 3.2.6.1, p. 259].

Lemma 4.5. *Let $s \in [0, 2]$ such that $s \neq \frac{1}{2}$ and $s \neq \frac{3}{2}$. Then the following generalisation of Hardy's Inequality holds*

$$\|d^{-s} g\|_{L^2(\Omega)} \leq C \|g\|_{H^s(\Omega)} \quad \text{for all } g \in H_0^s(\Omega). \tag{4.5}$$

We now prove Theorem 4.2.

Proof of Theorem 4.2. First, let denote by $e^{t\Delta}$ the semigroup generated by Δ in $L^2(\Omega)$ and notice that the solution u of (4.1) obeys

$$u(t) = e^{\Delta t} u_0 + L_T \left(\frac{1}{u^\delta} + f(x, u) \right) (t).$$

In the following we suppose that $0 < \eta < \frac{1}{1+\delta}$. If $0 \leq \delta < \frac{1}{2}$ then since $u \in \mathcal{C}$ we have $\frac{1}{u^\delta} = O(\frac{1}{d(x)^\delta})$ and it implies that $\frac{1}{u^\delta} + f(x, u) \in L^2(\Omega)$. Then using Lemma 4.4 we obtain $L_T(\frac{1}{u^\delta} + f(x, u)) \in C([0, T], H^{2-\eta}(\Omega))$. If $\frac{1}{2} \leq \delta < 1$, then from $u \in \mathcal{C}$ we have $\frac{1}{u^\delta} = O(\frac{1}{d(x)^\delta})$ and by Lemma 4.5 it implies that $\frac{1}{u^\delta} + f(x, u) \in C([0, T], (H_0^{\delta-\frac{1}{2}+\frac{\eta}{2}}(\Omega))') = C([0, T], (\mathcal{D}((-\Delta)^{\frac{\delta}{2}-\frac{1}{4}+\frac{\eta}{4}}))')$ (since $\frac{\delta}{2} - \frac{1}{4} + \frac{\eta}{4} \in (0, \frac{3}{4})$). Then Lemma 4.4 with $q = \frac{4}{\eta}$ gives $L_T(\frac{1}{u^\delta} + f(x, u)) \in C([0, T], H^{\frac{5}{2}-\delta-\eta}(\Omega))$. Next, if $\delta \geq 1$ then by Lemma 4.5 we have $\frac{1}{u^\delta} + f(x, u) \in C([0, T], (H_0^{\frac{2\delta}{\delta+1}-\frac{1}{2}+\frac{\eta}{2}}(\Omega))') = C([0, T], (\mathcal{D}((-\Delta)^{\frac{\delta}{\delta+1}-\frac{1}{4}+\frac{\eta}{4}}))')$ (since $\frac{\delta}{\delta+1} - \frac{1}{4} + \frac{\eta}{4} \in (0, \frac{3}{4})$). Therefore, using Lemma 4.4 with $q = \frac{2}{\eta}$ we get that $L_T(\frac{1}{u^\delta} + f(x, u)) \in C([0, T], H^{\frac{5}{2}-\frac{2\delta}{\delta+1}-\eta}(\Omega))$. As a consequence, if $u_0 \in X_\eta$ defined by

$$X_\eta = \begin{cases} H^{2-\eta}(\Omega) & \text{if } \delta < \frac{1}{2}, \\ H^{\frac{5}{2}-\delta-\eta}(\Omega) & \text{if } \frac{1}{2} \leq \delta < 1, \\ H^{\frac{5}{2}-\frac{2\delta}{\delta+1}-\eta}(\Omega) & \text{if } 1 \leq \delta, \end{cases} \tag{4.6}$$

then $t \rightarrow u(t) = e^{t\Delta} u_0 + L_T(\frac{1}{u^\delta} + f(x, u))(t) \in C([0, T], X_\eta)$. \square

Then Theorem 0.16 follows from Theorems 4.1 and 4.2. We end this section by proving Theorem 0.18.

Proof of Theorem 0.18. Let $\delta < 3$ and $u_0 \in H_0^1(\Omega) \cap \mathcal{C}$. Then, from Theorem 0.13, Theorem 0.15 and Theorem 0.16, the solution to (0.9) is unique, belongs to $C([0, +\infty), H_0^1(\Omega))$ and satisfies $\underline{u} \leq u \leq \bar{u}$ (with \underline{u}, \bar{u} as in the proof of Theorem 0.16), and

$$u(t) \rightarrow u_\infty \quad \text{in } L^\infty(\Omega) \text{ as } t \rightarrow +\infty,$$

where u_∞ is given by Theorem 0.8 with $p = 2$. To complete the proof of Theorem 0.18, let us show that

$$u(t) \rightarrow u_\infty \quad \text{in } H_0^1(\Omega) \text{ as } t \rightarrow +\infty. \tag{4.7}$$

For that, let $0 < \eta < \min(\frac{1}{2}, \frac{3-\delta}{2(1+\delta)})$ and note that X_η defined by (4.6) is compactly embedded in $H_0^1(\Omega)$. Moreover, for any $t > 0$, using Lemma 4.4 as in the proof of Theorem 4.2 and the fact that $u_0 \in \mathcal{C} \cap H_0^1(\Omega)$, we have for $t' \geq t$,

$$t' \rightarrow S(t')u_0 = e^{t'\Delta}u_0 + L_{t'}\left(\frac{1}{u^\delta} + f(x, u)\right) \in C([t, +\infty), X_\eta), \quad (4.8)$$

and since $\underline{u} \leq u \leq \bar{u}$,

$$\sup_{[t, +\infty]} \|u(t)\|_{X_\eta} < +\infty. \quad (4.9)$$

From the compactness embedding of X_η in $H_0^1(\Omega)$ together with (4.9), (4.7) follows. \square

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