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Local Controllability to Trajectories of the Magnetohydrodynamic Equations

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Abstract. In this paper we prove the local controllability to trajectories of the three dimensional magnetohydrodynamic equations by means of two internal controls, one in the velocity equations and the other in the magnetic field equations and both localized in an arbitrary small subset with not empty interior of the domain. This paper improves the previous results (Barbu et al. in Comm Pure Appl Math 56:732–783, 2003; Barbu et al. in Adv Differ Equ 10:481–504, 2005; Havârneanu et al. in Adv Differ Equ 11:893–929, 2006; Havârneanu, in SIAM J Control Optim 46:1802–1830, 2007) where the second control is not localized and it allows to deduce the local controllability to trajectories with boundary controls. The proof relies on the Carleman inequality for the Stokes system of Imanuvilov et al. (Carleman estimates for second order nonhomogeneous parabolic equations, preprint) to deal with the velocity equations and on a new Carleman inequality for a Dynamo-type equation to deal with the magnetic field equations.

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1. Introduction

In the present paper we are interested in proving the local controllability to trajectories of the following magnetohydrodynamic (MHD) equations:

$$\begin{cases} v_t - \Delta v + (v\cdot\nabla)v - (\operatorname{curl} B)\times B + \nabla p = \overline{f} + \mathbf{1}_{\mathcal{O}}u_1 & \text{in } Q,\\ B_t + \operatorname{curl}(\operatorname{curl} B) - \operatorname{curl}(v\times B) = \mathbf{1}_{\mathcal{O}}u_2 & \text{in } Q,\\ \operatorname{div} v = \operatorname{div} B = 0 & \text{in } Q,\\ v = \overline{v}, \quad B\cdot n = \overline{h} & \text{on } \Sigma,\\ (\operatorname{curl} B - v\times B)\times n = \overline{e} & \text{on } \Sigma,\\ v(0) = v_0, \quad B(0) = B_0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

The system (1.1) describes the motion of an incompressible and electrically conducting fluid in a bounded domain Ω of $\mathbb{R}^3$ during the time interval $(0,T)$ where T is a fixed positive time horizon. Above and in the following $\times$ denotes the vectorial product, $n(x)\in\mathbb{R}^3$ denotes the unit exterior normal vector field defined on the boundary $\partial\Omega$ which is supposed to be regular. Moreover, we use the notations $Q \overset{\text{def}}{=} (0,T)\times\Omega$ and $\Sigma \overset{\text{def}}{=} (0,T)\times\partial\Omega$ for cylindrical domains.

The fluid is characterized by its velocity $v(t,x)\in\mathbb{R}^3$, by a pressure function $p(t,x)\in\mathbb{R}$ and by the magnetic field $B(t,x)\in\mathbb{R}^3$. The system (1.1) is a coupling between the incompressible Navier–Stokes equations satisfied by the velocity and the induction equations satisfied by the magnetic field. These last are obtained from the Maxwell's equations and the coupling is due to Lorentz forces through the presence of terms $(\operatorname{curl} B)\times B$ and $v\times B$, we refer to [13] for a detailed explanation of the derivation of (1.1). Note that here, the usual non-dimensional constants that characterize the flow (Hartmann number, interaction parameter and magnetic Reynolds number) are supposed to be equal to one for simplicity.

In (1.1), $\overline{f}(t,x)\in\mathbb{R}^3$ is a given body force and $\overline{v}(t,x)\in\mathbb{R}^3$, $\overline{h}(t,x)\in\mathbb{R}$, $\overline{e}(t,x)\in\mathbb{R}^3$ are the boundary values of a given target solution $(\overline{v},\overline{B})$ which satisfies:

$$\begin{cases} \overline{v}_t - \Delta \overline{v} + (\overline{v} \cdot \nabla)\overline{v} - (\operatorname{curl} \overline{B}) \times \overline{B} + \nabla \overline{p} = \overline{f} & \text{in } Q, \\ \overline{B}_t + \operatorname{curl}(\operatorname{curl} \overline{B}) - \operatorname{curl}(\overline{v} \times \overline{B}) = 0 & \text{in } Q, \\ \operatorname{div} \overline{v} = \operatorname{div} \overline{B} = 0 & \text{in } Q, \\ \overline{B} \cdot n = \overline{h}, \quad (\operatorname{curl} \overline{B} - \overline{v} \times \overline{B}) \times n = \overline{e} & \text{on } \Sigma. \end{cases} \tag{1.2}$$

To be compatible with induction and divergence equalities in (1.2) these boundary values must satisfy:

$$\begin{gathered} \int_{\partial\Omega} \overline{v} \cdot n \mathrm{d}\Gamma = \int_{\partial\Omega} \overline{h} \mathrm{d}\Gamma = 0 \quad \text{on } (0, T), \\ \overline{h}_t = \operatorname{div}_\tau \overline{e} \quad \text{on } \Sigma, \end{gathered} \tag{1.3}$$

where div_τ is the surfacic divergence operator. The second line in (1.3) is obtained by taking the normal trace of the magnetic field equation. Note that the particular situation where $\overline{h} = 0$ and $\overline{e} = 0$ corresponds to a perfectly conductive boundary.

In (1.1), u_1, u_2 are control functions in $\mathbf{L}^2((0,T) \times \mathcal{O})$ which are used to reach the target state $(\overline{v}, \overline{B})$ at time T. Here $\mathcal{O}$ is a nonempty and simply connected open subset of Ω, $\mathbf{1}_{\mathcal{O}} : \mathbf{L}^2(\mathcal{O}) \to \mathbf{L}^2(\Omega)$ denotes the extension operator defined by $\mathbf{1}_{\mathcal{O}}(z)(x) = z(x)$ if $x \in \mathcal{O}$ and $\mathbf{1}_{\mathcal{O}}(z)(x) = 0$ else. Note that, above and in the following, we write in bold the spaces of vector-valued functions: $\mathbf{L}^2(\Omega) = (L^2(\Omega))^3$, $\mathbf{L}^2(\mathcal{O}) = (L^2(\mathcal{O}))^3$ etc. We underline that to guarantee the well-posedness of the controlled induction equations without an additional gradient of pressure we need to search a control u_2 satisfying

$$\begin{gathered} \operatorname{div} (\mathbf{1}_{\mathcal{O}} u_2) = 0 \ \text{ in } \ Q, \\ (\mathbf{1}_{\mathcal{O}} u_2) \cdot n = 0 \ \text{ on } \ \Sigma. \end{gathered} \tag{1.4}$$

For that, we choose u_2 of the form

$$u_2 = P_{\mathcal{O}} \tilde{u}_2 \tag{1.5}$$

where $\tilde{u}_2 \in \mathbf{L}^2((0,T) \times \mathcal{O})$ and $P_{\mathcal{O}}$ is the classical Helmholtz projector related to $\mathcal{O}$ (i.e the orthogonal projection operator from $\mathbf{L}^2(\mathcal{O})$ onto the completion of $\{v \in \mathbf{C}_c^\infty(\mathcal{O}) \mid \operatorname{div} v = 0 \text{ in } \mathcal{O}\}$ for the norm of $\mathbf{L}^2(\mathcal{O})$).

System (1.1) is said to be controllable to trajectories in time T, if for a given target solution $(\overline{v}, \overline{B})$ satisfying (1.2) and a given pair of initial data (v_0, B_0) there exist controls u_1, u_2 such that the solution (v, B) of (1.1) satisfies the terminal conditions:

$$v(T) = \overline{v}(T) \quad \text{ and } \quad B(T) = \overline{B}(T) \quad \text{ in } \Omega. \tag{1.6}$$

Such a property is said to be local if it holds for initial data v_0, B_0 close enough to $\overline{v}(0)$, $\overline{B}(0)$. It is this last property that we intend to prove in this paper.

From a practical point of view, the controllability to trajectories is an important notion because it implies the (open loop) stabilizability which is the first property to be checked in order to construct a stabilizing feedback control. For the stabilization of a parabolic system around a stationary state, a criterion weaker than exact controllability is known (see [5]) but if the target state is not stationary, prove the exact controllability seems to be the most reasonable alternative. For instance in [8], the design of a feedback control stabilizing the Navier–Stokes equations around an instationary state is deduced from the controllability properties of the Navier–Stokes equations.

Before to state our result, let us clarify our hypotheses. We assume that

$$\begin{gathered} \partial\Omega \text{ is a two dimensional manifold of class } C^{2,1} \text{ and is} \\ \text{composed of a finite number of connected components,} \end{gathered} \tag{1.7}$$

and that

$$\begin{gathered} \overline{v} \in L^2(0,T; \mathbf{H}^2(\Omega)) \cap H^1(0,T; \mathbf{L}^2(\Omega)), \\ \overline{B} \in L^2(0,T; \mathbf{H}^2(\Omega)) \cap H^1(0,T; \mathbf{L}^2(\Omega)). \end{gathered} \tag{1.8}$$

We also assume the following compatibility conditions for the initial data:

$$\begin{aligned} &\operatorname{div} v_0 = \operatorname{div} B_0 = 0 \ \text{ in } \Omega \\ &(v_0 - \overline{v}(0)) \cdot n = (B_0 - \overline{B}(0)) \cdot n = 0 \ \text{ on } \partial\Omega. \end{aligned} \tag{1.9}$$

Next, we denote by g_i, $i = 1, \dots, N$ a basis of the finite dimensional space:

$$X_N \overset{\text{def}}{=} \{y \in \mathbf{L}^2(\Omega) \mid \operatorname{div} y = 0 \text{ in } \Omega, \ \operatorname{curl} y = 0 \text{ in } \Omega, \ y \cdot n = 0 \text{ on } \partial\Omega\}. \tag{1.10}$$

The fact that the above space is finite dimensional is well-known, see for instance [12, Chap. IX] or [19] for a detailed characterization of X_N. We recall that if Ω is simply-connected we have $N = 0$ and X_N reduces to $\{0\}$ and if Ω is multiply-connected then $N \geq 1$ is the number of cuts required to make Ω simply-connected. Moreover, since $\mathcal{O}$ is simply connected we have $\mathrm{g}_i = \nabla p_i$ in $\mathcal{O}$ for some smooth function p_i and with (1.5) it yields:

$$\int_\Omega \mathbf{1}_{\mathcal{O}} u_2 \cdot \mathrm{g}_i \mathrm{d}x = \int_{\mathcal{O}} P_{\mathcal{O}} \tilde{u}_2 \cdot \nabla p_i \mathrm{d}x = 0. \tag{1.11}$$

As a consequence, if we multiply the second equality in (1.1) and in (1.2) by g_i and integrate over Ω, with an integration by parts we obtain

$$\frac{d}{dt}\int_\Omega B(t) \cdot \mathrm{g}_i \mathrm{d}x = -\int_{\partial\Omega} \overline{e}(t) \cdot \mathrm{g}_i \mathrm{d}\Gamma = \frac{d}{dt}\int_\Omega \overline{B}(t) \cdot \mathrm{g}_i \mathrm{d}x \tag{1.12}$$

which means that $\int_\Omega (B(t) - \overline{B}(t)) \cdot \mathrm{g}_i \mathrm{d}x$ must be constant. Then for the following we make the additional assumption

$$\forall i = 1, \dots N \quad \int_\Omega (B_0 - \overline{B}(0)) \cdot \mathrm{g}_i \mathrm{d}x = 0, \tag{1.13}$$

which implies the following condition for the magnetic field:

$$\forall i = 1, \dots N \quad \int_\Omega B(t) \cdot \mathrm{g}_i \mathrm{d}x = \int_\Omega \overline{B}(t) \cdot \mathrm{g}_i \mathrm{d}x \quad \text{ for a.e. } t \in (0, T). \tag{1.14}$$

Note that if we denote by $\mathcal{C}_i$, $i = 1, \dots, N$ a family of smooth cuts required to make Ω simply-connected then it is well-known (see [19, Lemme 1.4]) that (1.13) and (1.14) are respectively equivalent to $\int_{\mathcal{C}_i} (B_0 - \overline{B}(0)) \cdot n \mathrm{d}\Gamma = 0$ and

$$\int_{\mathcal{C}_i} B(t) \cdot n \mathrm{d}\Gamma = \int_{\mathcal{C}_i} \overline{B}(t) \cdot n \mathrm{d}\Gamma \quad \forall t \in (0, T), \quad \forall i = 1, \dots N.$$

We are now in position to state our main theorem. In what follows, $\mathbf{H}^{1/2}_{00}(\Omega)$ stands for the subspace of $\mathbf{H}^{1/2}_0(\Omega)$ composed with y such that $\int_\Omega \operatorname{dist}(x, \partial\Omega)^{-1} |y|^2 \, \mathrm{d}x < +\infty$.

Theorem 1.1. *Assume that Ω is a bounded domain of $\mathbb{R}^3$ and that (1.7), (1.8) are satisfied. There exists $c_0 > 0$ and $\varepsilon > 0$ such that for all $(v_0, B_0) \in \mathbf{H}^{1/2}_{00}(\Omega) \times \mathbf{H}^{1/2}(\Omega)$ satisfying (1.9), (1.13) and*

$$\|(v_0 - \overline{v}(0), B_0 - \overline{B}(0))\|_{\mathbf{H}^{1/2}_{00}(\Omega) \times \mathbf{H}^{1/2}(\Omega)} \leq \varepsilon$$

there exists $(u_1, u_2) \in \mathbf{L}^2((0, T) \times \mathcal{O}) \times \mathbf{L}^2((0, T) \times \mathcal{O})$ satisfying (1.4) such that system (1.1), (1.14) admits a solution

$$(v, B) \in L^2(0, T; \mathbf{H}^{3/2}(\Omega) \times \mathbf{H}^{3/2}(\Omega)) \cap C(0, T; \mathbf{H}^{1/2}(\Omega) \times \mathbf{H}^{1/2}(\Omega))$$

which satisfies the terminal condition (1.6) as well as the estimate:

$$\begin{aligned} &\|(v - \overline{v}, B - \overline{B})\|_{L^2(0,T;\mathbf{H}^{3/2}(\Omega) \times \mathbf{H}^{3/2}(\Omega))} + \|(v - \overline{v}, B - \overline{B})\|_{C(0,T;\mathbf{H}^{1/2}_{00}(\Omega) \times \mathbf{H}^{1/2}(\Omega))} \\ &\quad + \|(u_1, u_2)\|_{\mathbf{L}^2((0,T) \times \mathcal{O}) \times \mathbf{L}^2((0,T) \times \mathcal{O})} \\ &\quad \leq c_0 \|(v_0 - \overline{v}(0), B_0 - \overline{B}(0))\|_{\mathbf{H}^{1/2}_{00}(\Omega) \times \mathbf{H}^{1/2}(\Omega)}. \end{aligned}$$

Moreover, if $(\tilde{v},\tilde{B}) \in L^2(0,T;\mathbf{H}^{3/2}(\Omega)\times\mathbf{H}^{3/2}(\Omega))\cap C(0,T;\mathbf{H}^{1/2}(\Omega)\times\mathbf{H}^{1/2}(\Omega))$ *is another solution of* (1.1) *corresponding to the initial data* v_0, B_0 *and to the controls* u_1, u_2 *then* $(\tilde{v},\tilde{B})$ *and* (v,B) *must coincide i.e* $(\tilde{v},\tilde{B})\equiv(v,B)$.

Remark 1.2.
1. The main novelty of the above controllability result is that the two controls are localized in a small subset of the domain. This improves the results of [6,7,31,32] where only the control in the velocity equation is localized.
2. The regularity assumptions (1.8) are weaker than those made in previous work on the subject, see in particular [31,32]. Moreover, the initial regularity assumption $(v_0-\overline{v}(0), B_0-\overline{B}(0))\in\mathbf{H}^{1/2}_{00}(\Omega)\times\mathbf{H}^{1/2}(\Omega)$ is "optimal" in the sense that it is the weaker as possible to obtain a fixed-point solution as well as its uniqueness.
3. Although it follows from standard arguments, the second part of the theorem about the uniqueness of the controlled solution must be underlined because it guarantees that the action of the control is effective: there does not exist another "uncontrolled" trajectory (v,B) corresponding to internal body forces u_1, u_2 and which does not satisfy (1.6). Note that such a uniqueness property is not clear when dealing with density dependent fluid, see the forthcoming work [4].
4. If (1.13) is not satisfied then the terminal condition $B(T)=\overline{B}(T)$ cannot be obtained because (1.12) would imply that the equalities in (1.14) are violated at $t=T$. However, it is possible to get rid of (1.13) by adding a global control term in the induction equations. More precisely, an analogue version of Theorem 1.1 can be obtained without the assumption (1.13) and for a control with a global gradient term of the form
$$\mathbf{1}_{\mathcal{O}}u_2+\nabla\chi,\quad \chi\in L^2(0,T;H^1(\Omega)),$$
or for a control with a finite dimensional global term of the form
$$\mathbf{1}_{\mathcal{O}}P_{\mathcal{O}}u_2+\sum_{i=1}^{N}\alpha_i \mathrm{g}_i,\quad \alpha_i\in L^2(0,T),\quad i=1,\dots,N.$$
The adaptation of the proof is left to the reader.

The two dimensional counterpart of Theorem 1.1 is also true. If Ω is supposed to be only bounded in directions x_1, x_2 and invariant in direction x_3, if all functions only depend on x_1, x_2 and if each vector field has its third component equal to zero (and then is identified to its two first components i.e $v(x)={}^t(v_1(x_1,x_2),v_2(x_1,x_2),0)\equiv{}^t(v_1(x_1,x_2),v_2(x_1,x_2))$, etc), then we can identify Ω to its bounded two dimensional section and (1.1) reduces to:
$$\left\{\begin{array}{lr} v_t-\Delta v+(v\cdot\nabla)v+(\operatorname{curl}B)B^\perp+\nabla p=\overline{f}+\mathbf{1}_{\mathcal{O}}u_1 & \text{in } Q,\\ B_t+\operatorname{curl}(\operatorname{curl}B)-\operatorname{curl}(v\cdot B^\perp)=\mathbf{1}_{\mathcal{O}}u_2 & \text{in } Q,\\ \operatorname{div}v=\operatorname{div}B=0 & \text{in } Q,\\ v=\overline{v},\quad B\cdot n=\overline{h} & \text{on } \Sigma,\\ (\operatorname{curl}B-v\cdot B^\perp)=\overline{e} & \text{on } \Sigma,\\ v(0)=v_0,\quad B(0)=B_0 & \text{in } \Omega, \end{array}\right. \tag{1.15}$$
where we have used the notations ${}^t(a_1,a_2)^\perp\stackrel{\text{def}}{=}{}^t(a_2,-a_1)$, $\operatorname{curl}a\stackrel{\text{def}}{=}\partial_{x_1}a_2-\partial_{x_2}a_1$ for a vector field $a={}^t(a_1,a_2)$ and $\operatorname{curl}b\stackrel{\text{def}}{=}{}^t(\partial_{x_2}b,-\partial_{x_1}b)$ for a scalar function b. Then the following two dimensional version of Theorem 1.1 holds.

Theorem 1.3. *Assume that* Ω *is a bounded domain of* $\mathbb{R}^2$ *and that* (1.7), (1.8) *are satisfied. There exists* $c_0>0$ *and* $\varepsilon>0$ *such that for all* $(v_0,B_0)\in\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega)$ *satisfying* (1.9), (1.13) *and*
$$\|(v_0-\overline{v}(0),B_0-\overline{B}(0))\|_{\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega)}\le\varepsilon$$

there exists $(u_1, u_2) \in \mathbf{L}^2((0,T)\times\mathcal{O})\times\mathbf{L}^2((0,T)\times\mathcal{O})$ *satisfying* (1.4) *such that system* (1.15), (1.14) *admits a solution* $(v, B) \in L^2(0,T;\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega))\cap C(0,T;\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega))$ *which satisfies the terminal condition* (1.6) *as well as estimate*

$$\begin{aligned}\|(v-\overline{v}, B-\overline{B})\|_{L^2(0,T;\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega))} &+ \|(v-\overline{v}, B-\overline{B})\|_{C(0,T;\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega))}\\ &+ \|(u_1,u_2)\|_{\mathbf{L}^2((0,T)\times\mathcal{O})\times\mathbf{L}^2((0,T)\times\mathcal{O})}\\ &\le c_0\|(v_0-\overline{v}(0), B_0-\overline{B}(0))\|_{\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega)}.\end{aligned}$$

Moreover, if $(\tilde{v}, \tilde{B}) \in L^2(0,T;\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega))\cap C(0,T;\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega))$ *is another solution of* (1.15) *corresponding to the initial data* v_0*,* B_0 *and to the controls* u_1*,* u_2 *then* $(\tilde{v}, \tilde{B})$ *and* (v, B) *must coincide i.e* $(\tilde{v}, \tilde{B}) \equiv (v, B)$*.*

The proof of Theorem 1.3 is simpler than the one of Theorem 1.1, see Remark 4.4 below. Then in the following we omit it and we only consider the three dimensional case.

The local controllability of magnetohydrodynamic equations has been studied in [6,7,31,32]. However, these quoted works provide controllability results with a non local control of the form $P\mathbf{1}_{\mathcal{O}}u_2$ in the induction equations where u_2 does not necessarily satisfies (1.4) and where P is the Helmholtz projector related to Ω. This means that the control is of the form $\mathbf{1}_{\mathcal{O}}u_2+\nabla\chi$ in Q for some pressure function χ not necessarily supported in $\mathcal{O}$. As a consequence, these results cannot be used to deduce the controllability with a boundary control from an extension of the domain procedure. Such a drawback has already been pointed out in [5,39] when considering the feedback stabilizability of magnetohydrodynamic equations around a stationary solution. To deduce from above theorems the local controllability to trajectories with boundary control localized in an arbitrary small subset $\Gamma_c\subset\partial\Omega$, it suffices to extend Ω through Γ_c to some domain $\tilde{\Omega}$ and to apply Theorem 1.1 (or Theorem 1.3) to the MHD system extended to $\tilde{\Omega}$ for an internal control localized in $\mathcal{O}\subset\tilde{\Omega}\backslash\Omega$. The boundary control will then be obtained by taking the restriction to $(0,T)\times\Gamma_c$ of the controlled solution defined in $(0,T)\times\tilde{\Omega}$. We then obtain control functions (b_1, b_2, b_3) supported in Γ_c and satisfying

$$\begin{gathered}\int_{\partial\Omega} b_1\cdot n\,\mathrm{d}\Gamma = \int_{\partial\Omega} b_2\,\mathrm{d}\Gamma = 0 \quad \text{ on } (0,T),\\ (b_2)_t = \mathrm{div}_\tau b_3 \quad \text{ on } \Sigma,\end{gathered}\tag{1.16}$$

such that the conclusion of Theorem 1.1 (or of Theorem 1.3) remains true for system (1.1) with $u_1 = u_2 = 0$ and with the boundary data replaced by

$$\begin{aligned} v=\overline{v}+b_1,\quad B\cdot n &= \overline{h}+b_2 &&\text{on } \Sigma,\\ (\mathrm{curl}\,B - v\times B)\times n &= \overline{e}+b_3 &&\text{on } \Sigma.\end{aligned}$$

Here we follow a classical approach for proving the controllability of a parabolic system which consists in proving a global Carleman inequality for an adjoint system. It is a type of sophisticated weighted inequality which permits to bound the values of the solution in the whole domain in term of its values in a subdomain. Global Carleman inequalities were first introduced in [22–24] and then were extensively used to prove the exact controllability of Stokes-type system. About the exact controllability of Stokes and Navier–Stokes equations we refer to [10,11,17,18,20–22,29,34] about some other related coupled systems we refer to [16,26,28,30,37,41]. We must underline that our result deeply relies on a Carleman inequality for the Stokes system which has been recently proved in [36] (see also [3,35]).

To prove Theorem 1.1 we first remark that the local controllability to trajectories of (1.1), (1.14) is equivalent to the local null controllability of the system satisfied by $(w,\theta) = (v-\overline{v}, B-\overline{B})$. By taking (1.5) into account and renaming $\tilde{u}_2$ by u_2 for simplicity, the problem is then reduced to find (u_1, u_2) such that the solution (w, B) of

$$\begin{cases} w_t - \Delta w + (w\cdot\nabla)\overline{v} + (\overline{v}\cdot\nabla)w - (\overline{B}\cdot\nabla)\theta \\ \qquad -(\theta\cdot\nabla)\overline{B} - (\theta\cdot\nabla)\theta + (w\cdot\nabla)w + \nabla q = \mathbf{1}_{\mathcal{O}}u_1 & \text{in } Q, \\ \theta_t + \operatorname{curl}(\operatorname{curl}\theta) - \operatorname{curl}(w\times\overline{B} + \overline{v}\times\theta) \\ \qquad -\operatorname{curl}(w\times\theta) = \mathbf{1}_{\mathcal{O}}P_{\mathcal{O}}u_2 & \text{in } Q, \\ \operatorname{div} w = \operatorname{div}\theta = 0 & \text{in } Q, \\ \forall i = 1,\dots,N \quad \int_\Omega w\cdot \mathrm{g}_i \mathrm{d}x = 0 & \text{in } (0,T), \\ w = 0, \quad \theta\cdot n = 0 & \text{on } \Sigma, \\ (\operatorname{curl}\theta - \overline{v}\times\theta)\times n = 0 & \text{on } \Sigma, \\ w(0) = v_0 - \overline{v}(0), \quad \theta(0) = B_0 - \overline{B}(0) & \text{in } \Omega, \end{cases} \tag{1.17}$$

satisfies

$$w(T) = 0 \quad \text{and} \quad \theta(T) = 0 \quad \text{in } \Omega. \tag{1.18}$$

Note that to obtain that $(w,\theta) = (v-\overline{v}, B-\overline{B})$ satisfies (1.17) we have used the formula of vectorial analysis $(\operatorname{curl} a)\times a = (a\cdot\nabla)a - \nabla(|a|^2/2)$. In order to construct a solution to (1.17)–(1.18) with a fixed point procedure, a first step consists in replacing the nonlinear terms $(\theta\cdot\nabla)\theta - (w\cdot\nabla)w$ and $\operatorname{curl}(w\times\theta)$ by adequate nonhomogeneous terms g_1 and g_2 vanishing at time T and to construct a solution of the corresponding linear system:

$$\begin{cases} w_t - \Delta w + (w\cdot\nabla)\overline{v} + (\overline{v}\cdot\nabla)w - (\overline{B}\cdot\nabla)\theta \\ \qquad -(\theta\cdot\nabla)\overline{B} + \nabla q = g_1 + \mathbf{1}_{\mathcal{O}}u_1 & \text{in } Q, \\ \theta_t + \operatorname{curl}(\operatorname{curl}\theta) - \operatorname{curl}(w\times\overline{B} + \overline{v}\times\theta) = g_2 + \mathbf{1}_{\mathcal{O}}P_{\mathcal{O}}u_2 & \text{in } Q, \\ \operatorname{div} w = \operatorname{div}\theta = 0 & \text{in } Q, \\ \forall i = 1,\dots,N \quad \int_\Omega w\cdot \mathrm{g}_i \mathrm{d}x = 0 & \text{in } (0,T), \\ w = 0, \quad \theta\cdot n = 0 & \text{on } \Sigma, \\ (\operatorname{curl}\theta - \overline{v}\times\theta)\times n = 0 & \text{on } \Sigma, \\ w(0) = w_0, \quad \theta(0) = \theta_0 & \text{in } \Omega, \\ w(T) = 0, \quad \theta(T) = 0 & \text{in } \Omega. \end{cases} \tag{1.19}$$

For that, we use a classical duality argument which relies on a Carleman inequality for the adjoint system:

$$\begin{cases} -y_t - \Delta y - (D^s y)\overline{v} + (D^a\rho)\overline{B} + \nabla\pi = f_1 & \text{in } Q, \\ -\rho_t - \Delta\rho + (D^s y)\overline{B} - (D^a\rho)\overline{v} + \nabla\kappa + \sum_{i=1}^N \mu_i \mathrm{g}_i = f_2 & \text{in } Q, \\ \operatorname{div} y = \operatorname{div}\rho = 0 & \text{in } Q, \\ \forall i = 1,\dots,N \quad \int_\Omega \rho\cdot\mathrm{g}_i \mathrm{d}x = 0 & \text{in } (0,T), \\ y = 0, \quad \rho\cdot n = 0, \quad (\operatorname{curl}\rho)\times n = 0 & \text{on } \Sigma, \end{cases} \tag{1.20}$$

where we have used the notations $D^s y \stackrel{\text{def}}{=} \nabla y + {}^t\nabla y$ and $D^a y \stackrel{\text{def}}{=} \nabla y - {}^t\nabla y$. In (1.20), the pressure functions π and κ and the real values μ_i, $i = 1,\dots,N$ plays the role of the Lagrange multipliers related to the free divergence constraints and to conditions $\int_\Omega \rho\cdot\mathrm{g}_i\mathrm{d}x = 0$, $i = 1,\dots,N$.

To obtain a Carleman inequality for (1.20) we apply to the first equality the Carleman inequality for the Stokes system of [36] and we apply to the second equality a new Carleman inequality for a Dynamo-type equation that we prove in Sect. 4. We underline that unlike in [6,7,31,32] we do not treat directly the Dynamo-type equation but we prove a Carleman inequality for the curl of such equation, which has the advantage of getting rid of the terms $\nabla\kappa$ and $\mu_i g_i$. This last inequality is obtained from a Carleman inequality for a vectorial heat equation with an homogeneous Robin boundary condition on the normal component and with an homogeneous Dirichlet boundary condition on the tangential component. For that, we use the idea of [14] to get rid of boundary terms corresponding to the second Carleman parameter with odd exponents.

The rest of the paper is organized as follows. The Sect. 2 is dedicated to notations. The Sect. 3 is devoted to the statements of theorems related to each steps of the proof of Theorem 1.1, their proof are postponed to Sects. 4 and 5. The Sect. 4 concerns the proof of the Carleman inequality for the adjoint MHD system (1.20). In particular, we provide a Carleman inequality for a vectorial heat equation with an homogeneous Robin boundary condition on the normal component and with an homogeneous Dirichlet boundary condition on the tangential component. Section 5 is dedicated to more classical proofs: we deduce the existence of a solution of (1.19) from a duality argument and we deduce the local null controllability of the nonlinear MHD systems (1.17) from a fixed-point procedure. Finally, we postpone in an Appendix some regularity results for the Stokes system with a nonhomogeneous non standard boundary condition and for linear non autonomous magnetohydrodynamic equations.

2. Notations and Definitions

Let us recall that Ω is a bounded open subset of $\mathbb{R}^3$ with a regular boundary $\partial\Omega$ satisfying (1.7). We denote by $n = {}^t(n_1, n_2, n_3)$ the outward unit normal to $\partial\Omega$ (^{t} denotes the transpose). For a scalar function w or a vector field $y = {}^t(y_1, y_2, y_3)$ we define $\nabla w \stackrel{\text{def}}{=} {}^t(\partial_{x_1} w, \partial_{x_2} w, \partial_{x_3} w)$, $\nabla y \stackrel{\text{def}}{=} (\partial_{x_j} y_i)_{1\leq i,j\leq 3}$, ${}^t\nabla y \stackrel{\text{def}}{=} (\partial_{x_i} y_j)_{1\leq i,j\leq 3}$ and we use the notations $D^s y \stackrel{\text{def}}{=} \nabla y + {}^t\nabla y$ and $D^a y \stackrel{\text{def}}{=} \nabla y - {}^t\nabla y$. Moreover we define the normal derivatives $\frac{\partial w}{\partial n} \stackrel{\text{def}}{=} (\nabla w)\cdot n$ and $\frac{\partial y}{\partial n} \stackrel{\text{def}}{=} (\nabla y)n$ on $\partial\Omega$, we use the notations $y_n \stackrel{\text{def}}{=} y\cdot n$ and $y_\tau \stackrel{\text{def}}{=} y - y_n n$ for the normal and the tangential components of y on $\partial\Omega$ and we recall that $y_\tau = 0$ if and only if $y \times n = 0$ where $\times$ denotes the vectorial product. The divergence of y is defined by $\operatorname{div} y \stackrel{\text{def}}{=} \sum_{j=1}^3 \partial_{x_j} y_j$ and the curl of y is defined by

$$\operatorname{curl} y \stackrel{\text{def}}{=} \begin{pmatrix} \partial_{x_2} y_3 - \partial_{x_3} y_2 \\ \partial_{x_3} y_1 - \partial_{x_1} y_3 \\ \partial_{x_1} y_2 - \partial_{x_2} y_1 \end{pmatrix}.$$

We denote by $L^2(\Omega)$, $L^2(\partial\Omega)$, $H^r(\Omega)$, $H^r(\partial\Omega)$, $H^r_0(\Omega)$ and $H^{-r}(\Omega) = [H^r_0(\Omega)]'$ for $r \geq 0$, the usual Lebesgue and Sobolev spaces of scalar functions in Ω or in $\partial\Omega$, we define $H^{1/2}_{00}(\Omega)$ as the subspace of $H^{1/2}_0(\Omega)$ composed with functions y satisfying $\int_\Omega \operatorname{dist}(x, \partial\Omega)^{-1}|y|^2 \,\mathrm{d}x < +\infty$ (see [40, Thm. 11.7]), and we write in bold the spaces of vector-valued functions: $\mathbf{L}^2(\Omega) = (L^2(\Omega))^3$, $\mathbf{L}^2(\partial\Omega) = (L^2(\partial\Omega))^3$, etc. Above and in the following $[X]'$ stands for the dual space of X. Moreover, we define $\mathbf{H}^2_n(\Omega)$ as the space of functions $y \in \mathbf{H}^2(\Omega)$ such that $y\cdot n = 0$ on $\partial\Omega$ and for $r \in [0, 2]$ we define $\mathbf{H}^r_n(\Omega)$ as the interpolation space $[\mathbf{H}^2_n(\Omega), \mathbf{L}^2(\Omega)]_{1-r/2}$. Standard arguments guarantee that $\mathbf{H}^r_n(\Omega) = \mathbf{H}^r(\Omega)$ if $r \in [0, 1/2)$, that $\mathbf{H}^r_n(\Omega) = \{y \in \mathbf{H}^r(\Omega) \mid y\cdot n = 0 \text{ on } \partial\Omega\}$ if $r \in (1/2, 2]$ and that $\mathbf{H}^{1/2}_n(\Omega) = \{y \in \mathbf{H}^{1/2}(\Omega) \mid y\cdot n \in H^{1/2}_{00}(\Omega)\}$.

We also introduce the spaces of free divergence functions for $r \in [0, 2]$:

$$\mathbf{V}^r_n(\Omega) \stackrel{\text{def}}{=} \left\{ y \in \mathbf{H}^r(\Omega) \;\mid\; \operatorname{div} y = 0 \text{ in } \Omega, \quad y\cdot n = 0 \text{ on } \partial\Omega \right\},$$

$$\mathbf{V}^r_0(\Omega) \stackrel{\text{def}}{=} [\mathbf{V}^2_n(\Omega) \cap \mathbf{H}^1_0(\Omega), \mathbf{V}^0_n(\Omega)]_{1-r/2}, \quad \mathbf{V}^{-r}(\Omega) \stackrel{\text{def}}{=} [\mathbf{V}^r_0(\Omega)]',$$

and we recall that the orthogonal of $\mathbf{V}^0_n(\Omega)$ in $\mathbf{L}^2(\Omega)$ is given by:

$$[\mathbf{V}^0_n(\Omega)]^\perp = \left\{ \nabla p \mid p \in H^1(\Omega) \right\}. \tag{2.1}$$

Standard arguments guarantee that $\mathbf{V}_0^r(\Omega)=\mathbf{V}_n^r(\Omega)$ if $r\in[0,1/2)$, that $\mathbf{V}_0^r(\Omega)=\{y\in\mathbf{H}^r(\Omega)\cap\mathbf{V}_n^0(\Omega)\mid y=0 \text{ on } \partial\Omega\}$ if $r\in(\frac{1}{2},2]$ and that $\mathbf{V}_0^{1/2}(\Omega)=\mathbf{V}_n^{1/2}(\Omega)\cap\mathbf{H}_{00}^{1/2}(\Omega)$. Next, we recall that the family g_i, $i=1,\dots,N$ denotes a basis of the space X_N defined by (1.10), for $r\in[0,2]$ we define

$$\widehat{\mathbf{V}}_n^r(\Omega)\stackrel{\text{def}}{=}\left\{y\in\mathbf{V}_n^r(\Omega)\mid\int_\Omega y\cdot\mathrm{g}_i\mathrm{d}x=0\quad i=1,\dots,N\right\},$$

and we recall that the following Poincaré-type inequality holds

$$\forall y\in\widehat{\mathbf{V}}_n^1(\Omega)\qquad\int_\Omega|\mathrm{curl}\,y(t)|^2\mathrm{d}x\mathrm{d}\tau\geq c\int_\Omega(|\nabla y(t)|^2+|y(t)|^2)\mathrm{d}x,\tag{2.2}$$

where $c>0$ is a positive constant depending only on Ω, see [19, Lemme 1.6].

Moreover, the orthogonal of $\widehat{\mathbf{V}}_n^0(\Omega)$ in $\mathbf{L}^2(\Omega)$ is given by (see [19]):

$$[\widehat{\mathbf{V}}_n^0(\Omega)]^\perp=\left\{\nabla p+\sum_{i=1}^N\beta_i\mathrm{g}_i\mid p\in H^1(\Omega),\,\beta_i\in\mathbb{R},i=1,\dots,N\right\}.\tag{2.3}$$

Finally, we set $Q\stackrel{\text{def}}{=}(0,T)\times\Omega$, $\Sigma\stackrel{\text{def}}{=}(0,T)\times\partial\Omega$, for $r\geq0$ we define:

$$\mathbf{H}^{r,\frac{r}{2}}(Q)\stackrel{\text{def}}{=}L^2(0,T;\mathbf{H}^r(\Omega))\cap H^{\frac{r}{2}}(0,T;\mathbf{L}^2(\Omega)),$$
$$\mathbf{H}^{r,\frac{r}{2}}(\Sigma)\stackrel{\text{def}}{=}L^2(0,T;\mathbf{H}^r(\partial\Omega))\cap H^{\frac{r}{2}}(0,T;\mathbf{L}^2(\partial\Omega)),$$

and for two Hilbert spaces H_1 and H_2 we set:

$$W(0,T;H_1,H_2)\stackrel{\text{def}}{=}L^2(0,T;H_1)\cap H^1(0,T;H_2).$$

3. Main Steps of the Proof of Theorem 1.1

The proof of Theorem 1.1 relies on a global Carleman inequality for the adjoint system (1.20). Let us introduce some usual weight functions that appear in Carleman inequalities for parabolic systems. Let $\omega\subset\mathcal{O}$ a nonempty open subset such that $\overline{\omega}\subset\mathcal{O}$ and let $\eta\in C^2(\overline{\Omega})$ such that

$$\begin{aligned}\eta(x)=0&\quad\forall x\in\partial\Omega,\\ \eta(x)>0&\quad\forall x\in\Omega,\\ |\nabla\eta(x)|>0&\quad\forall x\in\overline{\Omega\backslash\omega}.\end{aligned}$$

For the existence of such a function see for instance [24] or [42, Appendix III]. Thus, we introduce $e\in C^\infty([0,1])$ such that $e(t)=t$ for $t\in(0,1/4)$, $e(t)=1-t$ for $t\in(3/4,1)$ and $e(t)\in[1/4,1/2]$ for $t\in(1/4,3/4)$, and we define $\ell(t)=Te(t/T)$. It is obvious to see that:

$$\ell\in C^\infty([0,T])\qquad\text{and}\qquad\begin{cases}\ell(t)=t&\forall t\in(0,T/4),\\ \ell(t)\in[T/4,T/2]&\forall t\in(T/4,3T/4),\\ \ell(t)=T-t&\forall t\in(3T/4,T).\end{cases}\tag{3.1}$$

Thus, for $k\geq2$ and $\lambda>1$ we define:

$$\begin{aligned}\alpha(t,x)&\stackrel{\text{def}}{=}\frac{e^{\lambda(\eta(x)+(k+1)\|\eta\|_\infty)}-e^{\lambda(k+2+\frac{1}{k})\|\eta\|_\infty}}{\ell(t)^k},\\ \varphi(t,x)&\stackrel{\text{def}}{=}\frac{e^{\lambda(\eta(x)+(k+1)\|\eta\|_\infty)}}{\ell(t)^k},\end{aligned}\tag{3.2}$$

and

$$\widehat{\alpha}(t) \stackrel{\text{def}}{=} \min_{x\in\overline{\Omega}} \alpha(t,x) = \frac{e^{\lambda(k+1)\|\eta\|_\infty} - e^{\lambda(k+2+\frac{1}{k})\|\eta\|_\infty}}{\ell(t)^k},$$
$$\widehat{\varphi}(t) \stackrel{\text{def}}{=} \min_{x\in\overline{\Omega}} \varphi(t,x) = \frac{e^{\lambda(k+1)\|\eta\|_\infty}}{\ell(t)^k}. \tag{3.3}$$

The above weight functions satisfy

$$T^{-1}|\ell(t)| + |(\ell)_t(t)| + T|(\ell)_{tt}(t)| \le c, \tag{3.4}$$

and for all $\lambda > 1$, $i = 1, 2, 3$ and $b \ge a$:

$$\lambda^{-2}|\partial^2_{x_i}\varphi(t,x)| + \lambda^{-1}|\partial_{x_i}\varphi(t,x)| \le c\varphi(t,x), \tag{3.5}$$
$$|\varphi^a(t,x)| \le cT^{(b-a)k}\varphi^b(t,x), \tag{3.6}$$
$$|\varphi_t(t,x)| + |\alpha_t(t,x)| \le c\varphi^{1+\frac{1}{k}}(t,x), \tag{3.7}$$
$$|\varphi_t(t,x)| + |\alpha_t(t,x)| \le cT^{k-1}\varphi^2(t,x), \tag{3.8}$$
$$|\alpha_{tt}(t,x)| \le cT^{2k-2}\varphi^3(t,x), \tag{3.9}$$

for some constant $c > 0$ which does not depend on T or λ.

The following theorem states a Carleman inequality for the adjoint system (1.20). Its proof is postponed to Sect. 4.

Theorem 3.1. *Assume $k \ge 4$. There exists $c_0 > 0$, $\lambda_0 > 0$ and $s_0 > 0$ such that for all $s \ge s_0$ and $\lambda \ge \lambda_0$ every solution of* (1.20) *satisfies:*

$$\begin{aligned}
&\int_Q e^{2s\alpha}|\nabla(\operatorname{curl} y)|^2 \mathrm{d}x\mathrm{d}\tau + s\lambda^2\int_Q e^{2s\alpha}\varphi|\nabla y|^2\mathrm{d}x\mathrm{d}\tau \\
&\quad + s^2\lambda^2\int_Q e^{2s\alpha}\varphi^2|\operatorname{curl} y|^2\mathrm{d}x\mathrm{d}\tau + s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|y|^2\mathrm{d}x\mathrm{d}\tau \\
&\quad + \lambda\int_Q e^{2s\alpha}|\nabla(\operatorname{curl}\rho)|^2\mathrm{d}x\mathrm{d}\tau + s^2\lambda^3\int_Q e^{2s\alpha}\varphi^2|\operatorname{curl}\rho|^2\mathrm{d}x\mathrm{d}\tau \\
&\le c_0\bigg(s\int_Q e^{2s\alpha}\varphi(|f_1|^2 + \lambda|f_2|^2)\mathrm{d}x\mathrm{d}\tau \\
&\quad + s^4\lambda^4\int_0^T\int_{\mathcal{O}} e^{2s\alpha}\varphi^4(|y|^2 + \lambda|P_{\mathcal{O}}\rho|^2)\mathrm{d}x\mathrm{d}\tau\bigg).
\end{aligned} \tag{3.10}$$

Next, we introduce $\tilde{\ell} \in C^\infty([0,T])$ such that $\tilde{\ell} \equiv \ell$ on $(\frac{T}{2}, T]$ and $\tilde{\ell} > 0$ on $[0, \frac{T}{2}]$ and we set

$$\widehat{\beta}(t) \stackrel{\text{def}}{=} \frac{e^{\lambda(k+1)\|\eta\|_\infty} - e^{\lambda(k+2+\frac{1}{k})\|\eta\|_\infty}}{\tilde{\ell}(t)^k}.$$

It is clear that for λ large enough we have:

$$\forall (t,x) \in \left[\frac{T}{2}, T\right] \times \overline{\Omega} \quad \alpha(t,x) < \frac{3}{4}\widehat{\beta}(t). \tag{3.11}$$

In the remaining part of this section we fix $k \ge 4$ and s, λ large enough so that (3.10) and (3.11) are satisfied.

The following theorem follows by combining the use of Theorem 6.11 in appendix and of Theorem 3.1 with (3.10), (3.11) and (2.2).

Theorem 3.2. *There exists $c > 0$ such that every solution of (1.20) satisfies:*

$$\begin{aligned}&\int_Q e^{2s\widehat{\beta}}(|\nabla y|^2 + |\nabla \rho|^2 + |y|^2 + |\rho|^2)\mathrm{d}x\mathrm{d}\tau + \|(y(0), \rho(0))\|_{\mathbf{V}^0_n(\Omega)\times\widehat{\mathbf{V}}^0_n(\Omega)}\\ &\quad \le c\bigg(\int_Q e^{s\frac{3\widehat{\beta}}{2}}(|f_1|^2 + |f_2|^2)\mathrm{d}x\mathrm{d}\tau + \int_0^T\int_{\mathcal{O}} e^{s\frac{3\widehat{\beta}}{2}}(|y|^2 + |P_{\mathcal{O}}\rho|^2)\mathrm{d}x\mathrm{d}\tau\bigg). \qquad (3.12)\end{aligned}$$

The proof of the above theorem consists in combining the use of estimate (6.20) in appendix on $(\psi y, \psi\rho)$, where ψ is a smooth nonnegative cut-off function such that $\psi \equiv 1$ on $(0, T/2)$ and $\psi \equiv 0$ on $(3T/4, T)$, and the use of the Carleman inequality (3.10) together with (3.11) and (2.2). Since it is very classical, we omit it and we refer to [17, Lemma 1] or [28, Lemma 2] for a completely similar proofs.

The Theorem 3.2 permits to obtain a solution of the nonhomogeneous controlled system (1.19). We introduce the following weight spaces:

$$\begin{aligned}\mathcal{G} &\overset{\text{def}}{=} \big\{(g_1, g_2) \in L^2(0, T; \mathbf{H}^{-1/2}(\Omega)) \times [\mathbf{H}^{1/2}_n(\Omega)]') \mid\\ &\qquad (e^{-s\widehat{\beta}}g_1, e^{-s\widehat{\beta}}g_2) \in L^2(0, T; \mathbf{V}^{-1/2}(\Omega) \times [\mathbf{H}^{1/2}_n(\Omega)]'\big\},\\ \mathcal{W} &\overset{\text{def}}{=} \big\{(w, \theta) \in W(0, T; \mathbf{V}^{3/2}_0(\Omega) \times \widehat{\mathbf{V}}^{3/2}_n(\Omega), \mathbf{V}^{-1/2}(\Omega) \times [\widehat{\mathbf{V}}^{1/2}_n(\Omega))]') \mid\\ &\qquad (e^{-s\frac{\widehat{\beta}}{2}}w, e^{-s\frac{\widehat{\beta}}{2}}\theta) \in W(0, T; \mathbf{V}^{3/2}_0(\Omega) \times \widehat{\mathbf{V}}^{3/2}_n(\Omega), \mathbf{V}^{-1/2}(\Omega) \times [\widehat{\mathbf{V}}^{1/2}_n(\Omega))]')\\ &\qquad (e^{-s\frac{3\widehat{\beta}}{2}}w, e^{-s\frac{3\widehat{\beta}}{2}}\theta) \in \mathbf{L}^2(Q) \times \mathbf{L}^2(Q)\big\}\\ \mathcal{U} &\overset{\text{def}}{=} \big\{(u_1, u_2) \in \mathbf{L}^2((0, T) \times \mathcal{O}) \times \mathbf{L}^2((0, T) \times \mathcal{O}) \mid\\ &\qquad (e^{-s\frac{3\widehat{\beta}}{2}}u_1, e^{-s\frac{3\widehat{\beta}}{2}}u_2) \in \mathbf{L}^2((0, T) \times \mathcal{O}) \times \mathbf{L}^2((0, T) \times \mathcal{O})\big\},\end{aligned}$$

equipped with the norms

$$\begin{aligned}\|(g_1, g_2)\|_{\mathcal{G}} &\overset{\text{def}}{=} \|(e^{-s\widehat{\beta}}g_1, e^{-s\widehat{\beta}}g_2)\|_{L^2((0,T);\mathbf{H}^{-1/2}(\Omega))\times[\mathbf{H}^{1/2}_n(\Omega))]')},\\ \|(w, \theta)\|_{\mathcal{W}} &\overset{\text{def}}{=} \|(e^{-s\frac{\widehat{\beta}}{2}}w, e^{-s\frac{\widehat{\beta}}{2}}\theta)\|_{W(0,T;\mathbf{V}^{3/2}_0(\Omega)\times\widehat{\mathbf{V}}^{3/2}_n(\Omega),\mathbf{V}^{-1/2}(\Omega)\times[\widehat{\mathbf{V}}^{1/2}_n(\Omega)]')}\\ &\qquad + \|(e^{-s\frac{3\widehat{\beta}}{2}}w, e^{-s\frac{3\widehat{\beta}}{2}}\theta)\|_{\mathbf{L}^2(Q)},\\ \|(u_1, u_2)\|_{\mathcal{U}} &\overset{\text{def}}{=} \|(e^{-s\frac{3\widehat{\beta}}{2}}u_1, e^{-s\frac{3\widehat{\beta}}{2}}u_2)\|_{\mathbf{L}^2((0,T)\times\mathcal{O})\times\mathbf{L}^2((0,T)\times\mathcal{O})}.\end{aligned}$$

We have the following null controllability result for the nonhomogeneous magnetohydrodynamic equations.

Theorem 3.3. *There exists $c_0 > 0$ such that for all $(w_0, \theta_0) \in \mathbf{V}^{1/2}_0(\Omega) \times \widehat{\mathbf{V}}^{1/2}_n(\Omega)$ and $(g_1, g_2) \in \mathcal{G}$ the control problem (1.19) admits a solution $(w, \theta, u_1, u_2) \in \mathcal{W} \times \mathcal{U}$ satisfying:*

$$\|(w, \theta)\|_{\mathcal{W}} + \|(u_1, u_2)\|_{\mathcal{U}} \le c_0(\|(g_1, g_2)\|_{\mathcal{G}} + \|(w_0, \theta_0)\|_{\mathbf{V}^{1/2}_0(\Omega)\times\widehat{\mathbf{V}}^{1/2}_n(\Omega)}).$$

Finally, the Theorem 1.1 is an obvious consequence of the following theorem which is obtained from Theorem 3.3 with a fixed-point argument.

Theorem 3.4. *There exist $c_0 > 0$ and $\varepsilon > 0$ such that for all $(w_0, \theta_0) \in \mathbf{V}^{1/2}_0(\Omega) \times \widehat{\mathbf{V}}^{1/2}_n(\Omega)$ such that*

$$\|(w_0, \theta_0)\|_{\mathbf{V}^{1/2}_0(\Omega)\times\widehat{\mathbf{V}}^{1/2}_n(\Omega)} \le \varepsilon$$

system (1.17)–(1.18) admits a solution $(w, \theta, u_1, u_2) \in \mathcal{W} \times \mathcal{U}$ satisfying:

$$\|(w, \theta)\|_{\mathcal{W}} + \|(u_1, u_2)\|_{\mathcal{U}} \le c_0\|(w_0, \theta_0)\|_{\mathbf{V}^{1/2}_0(\Omega)\times\widehat{\mathbf{V}}^{1/2}_n(\Omega)}.$$

Moreover, if $(\tilde{w}, \tilde{\theta}) \in C(0, T; \mathbf{H}^{1/2}(\Omega) \times \mathbf{H}^{1/2}(\Omega)) \cap L^2(0, T; \mathbf{H}^{3/2}(\Omega) \times \mathbf{H}^{3/2}(\Omega))$ is another solution of (1.17) corresponding to (w_0, θ_0) and (u_1, u_2) then $(\tilde{w}, \tilde{\theta})$ and (w, θ) must coincide, i.e $(\tilde{w}, \tilde{\theta}) \equiv (w, \theta)$.

4. A Carleman Inequality for the Adjoint System

4.1. Main Steps of the Proof of Theorem 3.1

We set $\zeta \overset{\text{def}}{=} \operatorname{curl}\rho$ and we apply the operator curl to the second equality in (1.20). By using the identities $(D^a\rho)\overline{B} = S(\overline{B})\operatorname{curl}\rho$ and $(D^a\rho)\overline{v} = S(\overline{v})\operatorname{curl}\rho$, where

$$S(z) \overset{\text{def}}{=} \begin{pmatrix} 0 & z_3 & -z_2 \\ -z_3 & 0 & z_1 \\ z_2 & -z_1 & 0 \end{pmatrix},$$

we get

$$\begin{cases} -y_t - \Delta y - (D^s y)\overline{v} + S(\overline{B})\zeta + \nabla\pi = f_1 & \text{in } Q, \\ -\zeta_t - \Delta\zeta + \operatorname{curl}\big((D^s y)\overline{B} - S(\overline{v})\zeta\big) = \operatorname{curl} f_2 & \text{in } Q, \\ \operatorname{div} y = \operatorname{div}\zeta = 0 & \text{in } Q, \\ y = \zeta\times n = 0 & \text{on } \Sigma. \end{cases} \tag{4.1}$$

Inequality (3.10) will be obtained by coupling the use of a Carleman inequality for Stokes equations and of a Carleman inequality for the curl of a Dynamo-type equations.

First, consider the adjoint Stokes equations with an homogeneous Dirichlet boundary condition:

$$\begin{cases} -\partial_t y - \Delta y + \nabla\pi = f & \text{in } Q, \\ \operatorname{div} y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma. \end{cases} \tag{4.2}$$

The following theorem can be found in [36] (see also [3,35]).

Theorem 4.1. *Assume $k \geq 4$. There exist $c_0 > 0$, $\lambda_0 > 0$ and $s_0 > 0$ such that for all $s \geq s_0$ and $\lambda \geq \lambda_0$ every solution of (4.2) satisfies*

$$\begin{aligned} &\int_Q e^{2s\alpha}|\nabla(\operatorname{curl} y)|^2 \mathrm{d}x\mathrm{d}\tau + s\lambda^2\int_Q e^{2s\alpha}\varphi|\nabla y|^2\mathrm{d}x\mathrm{d}\tau \\ &\quad + s^2\lambda^2\int_Q e^{2s\alpha}\varphi^2|\operatorname{curl} y|^2\mathrm{d}x\mathrm{d}\tau + s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|y|^2\mathrm{d}x\mathrm{d}\tau \\ &\quad \leq c_0\left(s\int_Q e^{2s\alpha}\varphi|f|^2\mathrm{d}x\mathrm{d}\tau + s^4\lambda^4\int_0^T\int_{\mathcal{O}} e^{2s\alpha}\varphi^4|y|^2\mathrm{d}x\mathrm{d}\tau\right). \end{aligned} \tag{4.3}$$

Next, consider the following equations:

$$\begin{cases} -\partial_t\zeta - \Delta\zeta = \operatorname{curl} f & \text{in } Q, \\ \operatorname{div}\zeta = 0 & \text{in } Q, \\ \zeta\times n = 0 & \text{on } \Sigma. \end{cases} \tag{4.4}$$

To prove a Carleman inequality for system (4.4) we use the trick of [38]. For $m > 0$ we set $\xi = e^{m\eta}\zeta$. Since $\operatorname{div}\zeta = 0$ in Q we deduce that $\operatorname{div}\xi = m\nabla\eta\cdot\xi$ in Q and combining this with the fact that $\operatorname{div}\xi|_{\partial\Omega} = \operatorname{div}_\tau\xi_\tau + (\operatorname{div} n)\xi_n + \frac{\partial\xi_n}{\partial n}$ on $\partial\Omega$ (where div_τ is the surfacic divergence operator) from $\xi_\tau = 0$ we deduce that ξ satisfies

$$\frac{\partial\xi_n}{\partial n} + \left(\operatorname{div} n - m\frac{\partial\eta}{\partial n}\right)\xi_n = 0 \text{ on } \Sigma.$$

Thus, if we set

$$\gamma = \operatorname{div} n - m\frac{\partial\eta}{\partial n} \tag{4.5}$$

and

$$g_1 = -2m\nabla\xi\nabla\eta - (me^{m\eta}\nabla\eta\times f + m^2|\nabla\eta|^2 - m\Delta\eta)\xi, \quad g_2 = e^{m\eta}f, \tag{4.6}$$

then ξ satisfies the following vectorial heat equation with a Robin boundary condition on ξ_n and with a Dirichlet boundary condition on ξ_τ:

$$\left\{\begin{array}{ll} -\partial_t\xi-\Delta\xi=g_1+\mathrm{curl}(g_2) & \text{in } Q,\\ \dfrac{\partial\xi_n}{\partial n}+\gamma\xi_n=0 & \text{on } \Sigma,\\ \xi_\tau=0 & \text{on } \Sigma.\end{array}\right. \tag{4.7}$$

The following Theorem states a Carleman inequality for system (4.7). Its proof is postponed to the Subsect. 4.2.

Theorem 4.2. *Assume $k\geq 2$ and suppose that $\min_{\partial\Omega}\gamma(x)>0$. There exist $c_0>0$, $\lambda_0>0$ and $s_0>0$ such that for all $s\geq s_0$ and $\lambda\geq\lambda_0$ every solution of (4.7) satisfies*

$$\begin{aligned} &s\lambda^2\int_Q e^{2s\alpha}\varphi|\nabla\xi|^2\mathrm{d}x\mathrm{d}\tau+s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|\xi|^2\mathrm{d}x\mathrm{d}\tau\\ &\quad\leq c_0\Big(\int_Q e^{2s\alpha}|g_1|^2\mathrm{d}x\mathrm{d}\tau+s^2\lambda^2\int_Q e^{2s\alpha}\varphi^2|g_2|^2\mathrm{d}x\mathrm{d}\tau\\ &\qquad+s^3\lambda^4\int_0^T\int_\omega e^{2s\alpha}\varphi^3|\xi|^2\mathrm{d}x\mathrm{d}\tau\Big).\end{aligned} \tag{4.8}$$

To deduce a Carleman inequality for system (4.4) from the above theorem we choose m large enough so that

$$-\frac{m}{2}\frac{\partial\eta}{\partial n}\leq\gamma\leq-2m\frac{\partial\eta}{\partial n}. \tag{4.9}$$

Theorem 4.3. *Assume $k\geq 2$. There exist $c_0>0$, $\lambda_0>0$ and $s_0>0$ such that for all $s\geq s_0$ and $\lambda\geq\lambda_0$ every solution of (4.4) satisfies*

$$\begin{aligned} &\lambda\int_Q e^{2s\alpha}|\nabla\zeta|^2\mathrm{d}x\mathrm{d}\tau+s^2\lambda^3\int_Q e^{2s\alpha}\varphi^2|\zeta|^2\mathrm{d}x\mathrm{d}\tau\\ &\quad\leq c_0\left(s\lambda\int_Q\varphi e^{2s\alpha}|f|^2\mathrm{d}x\mathrm{d}\tau+s^2\lambda^3\int_0^T\int_\omega e^{2s\alpha}\varphi^2|\zeta|^2\mathrm{d}x\mathrm{d}\tau\right).\end{aligned} \tag{4.10}$$

Proof. By applying (4.8) to (4.7)–(4.6) and recalling (4.9) we obtain

$$\begin{aligned} &s\lambda^2\int_Q e^{2s\alpha}\varphi|\nabla\xi|^2\mathrm{d}x\mathrm{d}\tau+s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|\xi|^2\mathrm{d}x\mathrm{d}\tau\\ &\quad\leq c_0\Big(\int_Q e^{2s\alpha}(m^2e^{2m\eta}|f|^2+m^4|\xi|^2+m^2|\nabla\xi|^2)\mathrm{d}x\mathrm{d}\tau\\ &\qquad+s^2\lambda^2\int_Q e^{2s\alpha}\varphi^2e^{2m\eta}|f|^2\mathrm{d}x\mathrm{d}\tau+s^3\lambda^4\int_0^T\int_\omega e^{2s\alpha}\varphi^3|\xi|^2\mathrm{d}x\mathrm{d}\tau\Big)\end{aligned}$$

and we conclude by choosing s large enough and recalling that $\zeta=e^{-m\eta}\xi$. □

Remark 4.4. The strategy of proof consisting in reducing (4.4) to system (4.7) is specific to the three dimensional case and proving Theorem 4.2 is useless in the two dimensional case. Indeed, in the two dimensional counterpart of (1.20) the boundary condition on $\mathrm{curl}\rho$ is simply $\mathrm{curl}\rho=0$ on Σ and (4.4) must be rewritten as

$$\left\{\begin{array}{ll} -\partial_t\zeta-\Delta\zeta=\mathrm{curl}f & \text{in } Q,\\ \mathrm{div}\,\zeta=0 & \text{in } Q,\\ \zeta=0 & \text{on } \Sigma.\end{array}\right. \tag{4.11}$$

Then inequality (4.10) follows from a direct application of the Carleman inequality for parabolic equations with source term in $H^{-1}(\Omega)$ proved in [33], see also [15, Lem. 2.1].

We are now in position to prove Theorem 3.1.

Proof of Theorem 3.1. If we apply (4.3) and (4.10) to the first and the second equality of (4.1) respectively and if we choose $\lambda \geq c(\|\overline{v}\|_{\mathbf{L}^\infty(Q)} + \|\overline{B}\|_{\mathbf{L}^\infty(Q)})$ and $c > 0$ large enough we obtain

$$\begin{aligned}
&\int_Q e^{2s\alpha}|\nabla(\operatorname{curl} y)|^2 \mathrm{d}x\mathrm{d}\tau + s\lambda^2 \int_Q e^{2s\alpha}\varphi|\nabla y|^2 \mathrm{d}x\mathrm{d}\tau \\
&\quad + s^2\lambda^2 \int_Q e^{2s\alpha}\varphi^2|\operatorname{curl} y|^2 \mathrm{d}x\mathrm{d}\tau + s^3\lambda^4 \int_Q e^{2s\alpha}\varphi^3|y|^2 \mathrm{d}x\mathrm{d}\tau \\
&\quad + \lambda \int_Q e^{2s\alpha}|\nabla(\operatorname{curl} \rho)|^2 \mathrm{d}x\mathrm{d}\tau + s^2\lambda^3 \int_Q e^{2s\alpha}\varphi^2|\operatorname{curl} \rho|^2 \mathrm{d}x\mathrm{d}\tau \\
&\leq C\Big(s \int_Q e^{2s\alpha}\varphi(|f_1|^2 + \lambda|f_2|^2)\mathrm{d}x\mathrm{d}\tau + s^4\lambda^4 \int_0^T \int_{\mathcal{O}} e^{2s\alpha}\varphi^3|y|^2 \mathrm{d}x\mathrm{d}\tau \\
&\quad + s^2\lambda^3 \int_0^T \int_\omega e^{2s\alpha}\varphi^2|\operatorname{curl} \rho|^2 \mathrm{d}x\mathrm{d}\tau \Big)
\end{aligned} \tag{4.12}$$

for $C > 0$ independent on s, λ. Next, we use a classical localization argument to obtain the second local term at the right of inequality (3.10). We recall that $\overline{\omega} \subset \mathcal{O}$ and let $\vartheta \in C_c^\infty(\mathcal{O})$, $0 \leq \vartheta \leq 1$ and $\vartheta \equiv 1$ in ω. First, we have:

$$s^2 \int_0^T \int_\omega e^{2s\alpha}\varphi^2|\operatorname{curl} \rho|^2 \mathrm{d}x\mathrm{d}\tau \leq s^2 \int_0^T \int_{\mathcal{O}} \vartheta e^{2s\alpha}\varphi^2|\operatorname{curl} \rho|^2 \mathrm{d}x\mathrm{d}\tau.$$

We recall that $P_{\mathcal{O}}\rho = \rho - \nabla\chi$ for some $\chi \in H^1(\Omega)$ and it implies $\operatorname{curl} \rho = \operatorname{curl} P_{\mathcal{O}}\rho$. Then with the following computations:

$$\begin{aligned}
s^2 \int_0^T \int_{\mathcal{O}} \vartheta e^{2s\alpha}\varphi^2|\operatorname{curl} \rho|^2 \mathrm{d}x\mathrm{d}\tau &= s^2 \int_0^T \int_{\mathcal{O}} \vartheta e^{2s\alpha}\varphi^2 \operatorname{curl} \rho \cdot \operatorname{curl} P_{\mathcal{O}}\rho \, \mathrm{d}x\mathrm{d}\tau \\
&= s^2 \int_0^T \int_{\mathcal{O}} \vartheta e^{2s\alpha}\varphi^2 \operatorname{curl}(\operatorname{curl} \rho) \cdot P_{\mathcal{O}}\rho \, \mathrm{d}x\mathrm{d}\tau \\
&\quad + s^2 \int_0^T \int_{\mathcal{O}} e^{2s\alpha}\varphi^2 (\nabla\vartheta \times (\operatorname{curl} \rho)) \cdot P_{\mathcal{O}}\rho \, \mathrm{d}x\mathrm{d}\tau \\
&\quad + 2\lambda s^2 \int_0^T \int_{\mathcal{O}} \vartheta e^{2s\alpha}\varphi^2 (\nabla\eta \times (\operatorname{curl} \rho)) \cdot P_{\mathcal{O}}\rho \, \mathrm{d}x\mathrm{d}\tau \\
&\quad + 2\lambda s^3 \int_0^T \int_{\mathcal{O}} \vartheta e^{2s\alpha}\varphi^3 (\nabla\eta \times (\operatorname{curl} \rho)) \cdot P_{\mathcal{O}}\rho \, \mathrm{d}x\mathrm{d}\tau,
\end{aligned}$$

we obtain with (3.6) and $\epsilon > 0$:

$$\begin{aligned}
s^2\lambda^3 \int_0^T \int_\omega e^{2s\alpha}\varphi|\operatorname{curl} \rho|^2 \mathrm{d}x\mathrm{d}\tau &\leq \epsilon\lambda \int_Q e^{2s\alpha}|\nabla(\operatorname{curl} \rho)|^2 \mathrm{d}x\mathrm{d}\tau \\
&\quad + \epsilon s^2\lambda^3 \int_Q e^{2s\alpha}\varphi^2|\operatorname{curl} \rho|^2 \mathrm{d}x\mathrm{d}\tau + \frac{C}{\epsilon} s^4\lambda^5 \int_0^T \int_{\mathcal{O}} e^{2s\alpha}\varphi^4|P_{\mathcal{O}}\rho|^2 \mathrm{d}x\mathrm{d}\tau.
\end{aligned}$$

Then with (4.12) and $\epsilon > 0$ small enough we get (3.10). □

4.2. A Carleman Inequality for a Vectorial Heat Equation with Mixed Robin and Dirichlet Boundary Conditions

The goal of this subsection is to prove Theorem 4.2. In the whole subsection C denotes a generic positive constant that may change from line to line, which depends on the geometry, but which is independent on T, s, λ. Moreover, we introduce the following boundary integral:

$$
\begin{aligned}
I_\Sigma(s,\lambda,\psi) \stackrel{\text{def}}{=} & -2s^3\lambda^3\int_\Sigma |\nabla\eta|^2\widehat{\varphi}^3\frac{\partial\eta}{\partial n}|\psi|^2\mathrm{d}\Sigma + 2s^2\lambda\int_\Sigma \widehat{\alpha}_t\widehat{\varphi}\frac{\partial\eta}{\partial n}|\psi|^2\mathrm{d}\Sigma\\
& -4s\lambda\int_\Sigma \widehat{\varphi}\frac{\partial\eta}{\partial n}\left|\frac{\partial\psi}{\partial n}\right|^2\mathrm{d}\Sigma + 2s\lambda\int_\Sigma \widehat{\varphi}\frac{\partial\eta}{\partial n}\left|\nabla\psi\right|^2\mathrm{d}\Sigma\\
& -4s\lambda^2\int_\Sigma \left|\frac{\partial\eta}{\partial n}\right|^2\widehat{\varphi}\frac{\partial\psi}{\partial n}\psi\mathrm{d}\Sigma + 2\int_\Sigma \frac{\partial\psi}{\partial n}\psi_t\mathrm{d}\Sigma.
\end{aligned}
$$

We will need the following Carleman inequality for the heat equation with a nonhomogeneous boundary condition.

Lemma 4.5. *There exist $\lambda_0 > 0$, $c_0 > 0$ and $c_1 > 0$ independent on T and such that for all $\lambda \geq \lambda_0$ and $s \geq c_1(T^k + T^{k-1})$ the following inequality holds for all $q \in C^2(\overline{Q})$:*

$$
\begin{aligned}
& s^{-1}\int_Q e^{2s\alpha}\varphi^{-1}(|\partial_t q|^2 + |\Delta q|^2)\mathrm{d}x\mathrm{d}\tau + s\lambda^2\int_Q e^{2s\alpha}\varphi|\nabla q|^2\mathrm{d}x\mathrm{d}\tau\\
& \quad + s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|q|^2\mathrm{d}x\mathrm{d}\tau + I_\Sigma(s,\lambda,e^{s\alpha}q) \qquad (4.13)\\
& \quad \leq c_0\left(\int_Q e^{2s\alpha}|\partial_t q + \Delta q|^2\mathrm{d}x\mathrm{d}\tau + s^3\lambda^4\int_0^T\int_\omega e^{2s\alpha}\varphi^3|q|^2\mathrm{d}x\mathrm{d}\tau\right).
\end{aligned}
$$

For the proof of Lemma 4.5 we refer to the Appendix of [14].

Next, to get rid of the boundary terms in (4.13) corresponding to powers of λ with odd exponents, we need another Carleman inequality involving weight functions where η is replaced by $-\eta$. For $k \geq 2$ and $\lambda > 1$ we define:

$$
\begin{aligned}
\tilde{\alpha}(t,x) &= \frac{e^{\lambda(-\eta(x)+(k+1)\|\eta\|_\infty)} - e^{\lambda(k+2+\frac{1}{k})\|\eta\|_\infty}}{\ell(t)^k},\\
\tilde{\varphi}(t,x) &= \frac{e^{\lambda(-\eta(x)+(k+1)\|\eta\|_\infty)}}{\ell(t)^k}.
\end{aligned}
\qquad (4.14)
$$

We verify that $\tilde{\varphi}$ satisfies (3.5)–(3.6) and $\tilde{\alpha}$ satisfies

$$
\begin{aligned}
|\tilde{\alpha}_t(t,x)| &\leq Ce^{\lambda(1+\frac{1}{k})\|\eta\|_\infty}\tilde{\varphi}^{1+\frac{1}{k}}(t,x),\\
|\tilde{\alpha}_{tt}(t,x)| &\leq Ce^{\lambda\frac{1}{k}\|\eta\|_\infty}\tilde{\varphi}^{1+\frac{2}{k}}(t,x),\\
|\tilde{\alpha}_t(t,x)| &\leq Ce^{\lambda(1+\frac{1}{k})\|\eta\|_\infty}T^{k-1}\tilde{\varphi}^2(t,x),\\
|\tilde{\alpha}_{tt}(t,x)| &\leq Ce^{\lambda\frac{1}{k}\|\eta\|_\infty}T^{2k-2}\tilde{\varphi}^3(t,x).
\end{aligned}
$$

Note that we have

$$
\tilde{\alpha}(t,x) = \alpha(t,x) = \widehat{\alpha}(t) \quad \text{and} \quad \tilde{\varphi}(t,x) = \varphi(t,x) = \widehat{\alpha}(t) \quad \forall x \in \partial\Omega. \qquad (4.15)
$$

The proof of the following lemma can also be found in the Appendix of [14].

Lemma 4.6. *There exist $\lambda_0 > 0$, $c_0 > 0$ and $c_1 > 0$ independent on T and such that for all $\lambda \geq \lambda_0$ and $s \geq c_1 e^{\lambda(1+\frac{1}{k})\|\eta\|_\infty}(T^k + T^{k-1})$ the following inequality holds for all $q \in C^2(\overline{Q})$:*

$$
\begin{aligned}
& s^{-1}\int_Q e^{2s\tilde{\alpha}}\tilde{\varphi}^{-1}(|\partial_t q|^2 + |\Delta q|^2)\mathrm{d}x\mathrm{d}\tau + s\lambda^2\int_Q e^{2s\tilde{\alpha}}\tilde{\varphi}|\nabla q|^2\mathrm{d}x\mathrm{d}\tau\\
& \quad + s^3\lambda^4\int_Q e^{2s\tilde{\alpha}}\tilde{\varphi}^3|q|^2\mathrm{d}x\mathrm{d}\tau + I_\Sigma(s,-\lambda,e^{s\tilde{\alpha}}q)\\
& \quad \leq c_0\left(\int_Q e^{2s\tilde{\alpha}}|\partial_t q + \Delta q|^2\mathrm{d}x\mathrm{d}\tau + s^3\lambda^4\int_0^T\int_\omega e^{2s\tilde{\alpha}}\tilde{\varphi}^3|q|^2\mathrm{d}x\mathrm{d}\tau\right). \qquad (4.16)
\end{aligned}
$$

By combining the two previous lemmas we get the following theorem.

Theorem 4.7. *There exist $\lambda_0 > 0$, $c_0 > 0$ and $c_1 > 0$ independent on T and such that for all $\lambda \geq \lambda_0$ and $s \geq c_1 e^{\lambda(1+\frac{1}{k})\|\eta\|_\infty}(T^k + T^{k-1})$ the following inequality holds for all $q \in C^2(\overline{Q})$:*

$$\begin{aligned}
& s^{-1} \int_Q e^{2s\alpha} \varphi^{-1} (|\partial_t q|^2 + |\Delta q|^2) \mathrm{d}x\mathrm{d}\tau \\
& \quad + s\lambda^2 \int_Q e^{2s\alpha} \varphi |\nabla q|^2 \mathrm{d}x\mathrm{d}\tau + s^3\lambda^4 \int_Q e^{2s\alpha}\varphi^3 |q|^2 \mathrm{d}x\mathrm{d}\tau \\
& \leq c_0 \left(\int_Q e^{2s\alpha} |\partial_t q + \Delta q|^2 \mathrm{d}x\mathrm{d}\tau + s^3\lambda^4 \int_0^T \int_\omega e^{2s\alpha}\varphi^3 |q|^2 \mathrm{d}x\mathrm{d}\tau \right) \\
& \quad + 8s\lambda^2 \int_\Sigma e^{2s\widehat{\alpha}} \widehat{\varphi} \left| \frac{\partial \eta}{\partial n} \right|^2 (1 + s\widehat{\varphi}) \frac{\partial q}{\partial n} q \mathrm{d}\Sigma - 4 \int_\Sigma e^{s\widehat{\alpha}} \frac{\partial q}{\partial n} (e^{s\widehat{\alpha}} q)_t \mathrm{d}\Sigma.
\end{aligned}$$

Proof. We add (4.13) and (4.16) and since $\tilde{\alpha} \leq \alpha$ and $\tilde{\varphi} \leq \varphi$ we first get

$$\begin{aligned}
& s^{-1} \int_Q e^{2s\alpha} \varphi^{-1} (|\partial_t q|^2 + |\Delta q|^2) \mathrm{d}x\mathrm{d}\tau + s\lambda^2 \int_Q e^{2s\alpha} \varphi |\nabla q|^2 \mathrm{d}x\mathrm{d}\tau \\
& \quad + s^3\lambda^4 \int_Q e^{2s\alpha}\varphi^3 |q|^2 \mathrm{d}x\mathrm{d}\tau + I_\Sigma(s, \lambda, e^{s\alpha} q) + I_\Sigma(s, -\lambda, e^{s\tilde{\alpha}} q) \\
& \leq C \left(\int_Q e^{2s\alpha} |\partial_t q + \Delta q|^2 \mathrm{d}x\mathrm{d}\tau + s^3\lambda^4 \int_0^T \int_\omega e^{2s\alpha}\varphi^3 |q|^2 \mathrm{d}x\mathrm{d}\tau \right),
\end{aligned}$$

for $\lambda \geq \lambda_0$ and $s \geq Ce^{\lambda(1+\frac{1}{k})\|\eta\|_\infty}(T^k + T^{k-1})$. Thus, from (4.15) we deduce that $\nabla(e^{s\alpha} q) = e^{s\widehat{\alpha}}(s\lambda\widehat{\varphi}\nabla\eta q + \nabla q)$, $\nabla(e^{s\tilde{\alpha}} q) = e^{s\widehat{\alpha}}(-s\lambda\widehat{\varphi}\nabla\eta q + \nabla q)$ and that $(e^{s\alpha} q)_t = (e^{s\tilde{\alpha}} q)_t = (e^{s\widehat{\alpha}} q)_t$ on $\partial\Omega$, and we conclude with the calculations:

$$\begin{aligned}
& I_\Sigma(s, \lambda, e^{s\alpha} q) + I_\Sigma(s, -\lambda, e^{s\tilde{\alpha}} q) \\
& = -4s\lambda \int_\Sigma e^{2s\widehat{\alpha}} \widehat{\varphi} \frac{\partial \eta}{\partial n} \left(\left| \frac{\partial q}{\partial n} + s\lambda\widehat{\varphi} \frac{\partial \eta}{\partial n} q \right|^2 - \left| \frac{\partial q}{\partial n} - s\lambda\widehat{\varphi} \frac{\partial \eta}{\partial n} q \right|^2 \right) \mathrm{d}\Sigma \\
& \quad + 2s\lambda \int_\Sigma e^{2s\widehat{\alpha}} \widehat{\varphi} \frac{\partial \eta}{\partial n} \left(|\nabla q + s\lambda\widehat{\varphi}\nabla\eta q|^2 - |\nabla q - s\lambda\widehat{\varphi}\nabla\eta q|^2 \right) \mathrm{d}\Sigma \\
& \quad - 4s\lambda^2 \int_\Sigma e^{2s\widehat{\alpha}} \left| \frac{\partial \eta}{\partial n} \right|^2 \widehat{\varphi} \left(\frac{\partial q}{\partial n} + s\lambda\widehat{\varphi} \frac{\partial \eta}{\partial n} q \right) q \mathrm{d}\Sigma \\
& \quad - 4s\lambda^2 \int_\Sigma e^{2s\widehat{\alpha}} \left| \frac{\partial \eta}{\partial n} \right|^2 \widehat{\varphi} \left(\frac{\partial q}{\partial n} - s\lambda\widehat{\varphi} \frac{\partial \eta}{\partial n} q \right) q \mathrm{d}\Sigma \\
& \quad + \int_\Sigma e^{s\widehat{\alpha}} \left(\frac{\partial q}{\partial n} + s\lambda\widehat{\varphi} \frac{\partial \eta}{\partial n} q \right) (e^{s\widehat{\alpha}} q)_t \mathrm{d}\Sigma \\
& \quad + 2 \int_\Sigma e^{s\widehat{\alpha}} \left(\frac{\partial q}{\partial n} - s\lambda\widehat{\varphi} \frac{\partial \eta}{\partial n} q \right) (e^{s\widehat{\alpha}} q)_t \mathrm{d}\Sigma \\
& = -8s^2\lambda^2 \int_\Sigma e^{2s\widehat{\alpha}} \widehat{\varphi}^2 \left| \frac{\partial \eta}{\partial n} \right|^2 \frac{\partial q}{\partial n} q \mathrm{d}\Sigma - 8s\lambda^2 \int_\Sigma e^{2s\widehat{\alpha}} \widehat{\varphi} \left| \frac{\partial \eta}{\partial n} \right|^2 \frac{\partial q}{\partial n} q \mathrm{d}\Sigma \\
& \quad + 4 \int_\Sigma e^{s\widehat{\alpha}} \frac{\partial q}{\partial n} (e^{s\widehat{\alpha}} q)_t \mathrm{d}\Sigma.
\end{aligned}$$

□

A consequence of Theorem 4.7 is the following Carleman inequality for a vectorial heat equation with mixed Robin and Dirichlet boundary conditions.

Corollary 4.8. *There exist $\lambda_0 > 0$, $c_0 > 0$ and $c_1 > 0$ independent on T and such that for all $\lambda \geq \lambda_0$ and $s \geq c_1 e^{\lambda(1+\frac{1}{k})\|\eta\|_\infty}(T^k + T^{k-1})$ the following inequality holds for all $\xi \in \mathbf{C}^2(\overline{Q})$ such that $\xi_\tau = 0$ and $\frac{\partial \xi_n}{\partial n} + \gamma \xi_n = 0$ on Σ:*

$$
\begin{aligned}
&s^{-1}\int_Q e^{2s\alpha}\varphi^{-1}(|\partial_t\xi|^2 + |\Delta\xi|^2)\mathrm{d}x\mathrm{d}\tau + s\lambda^2\int_Q e^{2s\alpha}\varphi|\nabla\xi|^2\mathrm{d}x\mathrm{d}\tau\\
&\quad + s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|\xi|^2\mathrm{d}x\mathrm{d}\tau + s^2\lambda^2\int_\Sigma e^{2s\widehat{\alpha}}\widehat{\varphi}^2\gamma|\xi_n|^2\mathrm{d}\Sigma\\
&\quad \leq c_0\bigg(\int_Q e^{2s\alpha}|\partial_t\xi + \Delta\xi|^2\mathrm{d}x\mathrm{d}\tau + s^3\lambda^4\int_0^T\int_\omega e^{2s\alpha}\varphi^3|\xi|^2\mathrm{d}x\mathrm{d}\tau\bigg).
\end{aligned}
$$

Proof. By applying Theorem 4.7 to each component of ξ we obtain:

$$
\begin{aligned}
&s^{-1}\int_Q e^{2s\alpha}\varphi^{-1}(|\partial_t\xi|^2 + |\Delta\xi|^2)\mathrm{d}x\mathrm{d}\tau\\
&\quad +s\lambda^2\int_Q e^{2s\alpha}\varphi|\nabla\xi|^2\mathrm{d}x\mathrm{d}\tau + s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|\xi|^2\mathrm{d}x\mathrm{d}\tau\\
&\quad \leq C\bigg(\int_Q e^{2s\alpha}|\partial_t\xi + \Delta\xi|^2\mathrm{d}x\mathrm{d}\tau + s^3\lambda^4\int_0^T\int_\omega e^{2s\alpha}\varphi^3|\xi|^2\mathrm{d}x\mathrm{d}\tau\bigg)\\
&\quad +4s\lambda^2\int_\Sigma e^{2s\widehat{\alpha}}\widehat{\varphi}\left|\frac{\partial\eta}{\partial n}\right|^2(1+s\widehat{\varphi})\frac{\partial\xi}{\partial n}\cdot\xi\mathrm{d}\Sigma - 2\int_\Sigma e^{s\widehat{\alpha}}\frac{\partial\xi}{\partial n}\cdot(e^{s\widehat{\alpha}}\xi)_t\mathrm{d}\Sigma,
\end{aligned}
\tag{4.17}
$$

and the conclusion follows from the fact that $\xi_\tau = 0$ and $\frac{\partial\xi_n}{\partial n} + \gamma\xi_n = 0$ yield

$$
\int_\Sigma e^{s\widehat{\alpha}}\frac{\partial\xi}{\partial n}\cdot(e^{s\widehat{\alpha}}\xi)_t\mathrm{d}\Sigma = -\tfrac{1}{2}\int_\Sigma \gamma((e^{s\widehat{\alpha}}\xi_n)^2)_t\mathrm{d}\Sigma = 0,
$$

and from the fact that $\frac{\partial\xi}{\partial n}\cdot\xi = -\gamma|\xi_n|^2$ yields

$$
4s\lambda^2\int_\Sigma e^{2s\widehat{\alpha}}\widehat{\varphi}\left|\frac{\partial\eta}{\partial n}\right|^2(1+s\widehat{\varphi})\frac{\partial\xi}{\partial n}\cdot\xi\mathrm{d}\Sigma \leq -Cs\lambda^2\int_\Sigma e^{2s\widehat{\alpha}}\widehat{\varphi}(1+s\widehat{\varphi})\gamma|\xi_n|^2\mathrm{d}\Sigma.
$$

□

Finally, by duality we obtain a Carleman inequality for (4.7).

Theorem 4.9. *There exist $\lambda_0 > 0$, $c_0 > 0$ and $c_1 > 0$ independent on T and such that for all $\lambda \geq \lambda_0$ and $s \geq c_1 e^{\lambda(1+\frac{1}{k})\|\eta\|_\infty}(T^k + T^{k-1})$ every solution of (4.7) satisfies*

$$
\begin{aligned}
&s\lambda^2\int_Q e^{2s\alpha}\varphi|\nabla\xi|^2\mathrm{d}x\mathrm{d}\tau + s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|\xi|^2\mathrm{d}x\mathrm{d}\tau\\
&\quad \leq c_0\bigg(\int_Q e^{2s\alpha}|g_1|^2\mathrm{d}x\mathrm{d}\tau + s^2\lambda^2\int_Q e^{2s\alpha}\varphi^2|g_2|^2\mathrm{d}x\mathrm{d}\tau\\
&\quad +s^3\lambda^4\int_0^T\int_\omega e^{2s\alpha}\varphi^3|\xi|^2\mathrm{d}x\mathrm{d}\tau\bigg).
\end{aligned}
\tag{4.18}
$$

Proof. The proof follows from a now standard duality argument.

First, we construct a solution of

$$
\left\{
\begin{aligned}
&\partial_t z - \Delta z = s^3\lambda^4 e^{2s\alpha}\varphi^3\xi + \mathbf{1}_\omega u && \text{in } Q,\\
&\frac{\partial z_n}{\partial n} + \gamma z_n = 0 && \text{on } \Sigma,\\
&z_\tau = 0 && \text{on } \Sigma,\\
&z(0) = z(T) = 0 && \text{on } \Omega,
\end{aligned}
\right.
\tag{4.19}
$$

satisfying for $s \geq C(T^k + T^{k-1})$ and $\lambda > \lambda_0$, and $C > 0$, $\lambda_0 > 0$ large enough:

$$\begin{aligned}\int_Q e^{-2s\alpha}|z|^2 \mathrm{d}x\mathrm{d}\tau + s^{-3}\lambda^{-4}\int_0^T\int_\omega e^{-2s\alpha}\varphi^{-3}|u|^2\mathrm{d}x\mathrm{d}\tau \\ + s^{-2}\lambda^{-2}\int_Q e^{-2s\alpha}\varphi^{-2}|\nabla z|^2\mathrm{d}x\mathrm{d}\tau \leq Cs^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|\xi|^2\mathrm{d}x\mathrm{d}\tau.\end{aligned} \tag{4.20}$$

For that we set

$$\mathcal{P}_0 \overset{\text{def}}{=} \left\{ w \in \mathbf{C}^2(\overline{Q}) \mid \frac{\partial w_n}{\partial n} + \gamma w_n = 0 \text{ and } w_\tau = 0 \text{ on } \Sigma \right\},$$

and we consider the bilinear form $b(\cdot,\cdot) : \mathcal{P}_0 \times \mathcal{P}_0 \to \mathbb{R}$ defined by

$$\begin{aligned}b(w,w') = \int_Q e^{2s\alpha}(\partial_t w + \Delta w)\cdot(\partial_t w' + \Delta w')\mathrm{d}x\mathrm{d}\tau \\ + s^3\lambda^4\int_0^T\int_\omega e^{2s\alpha}\varphi^3 w\cdot w'\mathrm{d}x\mathrm{d}\tau,\end{aligned}$$

as well as the following linear form on $\mathcal{P}_0$:

$$l(w') = s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3\xi\cdot w'\mathrm{d}x\mathrm{d}\tau.$$

The Corollary 4.8 guarantees that the semi-norm induced by $b(\cdot,\cdot)$ is a norm. If $\mathcal{P}$ is the completion of $\mathcal{P}_0$ for this norm, then Corollary 4.8 guarantees the continuity of l on $\mathcal{P}$, and from the Lax-Milgram Lemma we obtain a unique solution $w \in \mathcal{P}$ to

$$b(w,w') = l(w'), \quad \forall w' \in \mathcal{P}. \tag{4.21}$$

Moreover, w satisfies

$$\begin{aligned}\int_Q e^{2s\alpha}|\partial_t w + \Delta w|^2\mathrm{d}x\mathrm{d}\tau + s^3\lambda^4\int_0^T\int_\omega e^{2s\alpha}\varphi^3|w|^2\mathrm{d}x\mathrm{d}\tau \\ \leq Cs^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|\xi|^2\mathrm{d}x\mathrm{d}\tau.\end{aligned} \tag{4.22}$$

If we set $z = e^{2s\alpha}(\partial_t w + \Delta w)$ and $u = -s^3\lambda^4 e^{2s\alpha}\varphi^3 w$ then (4.21) becomes:

$$\int_Q z\cdot(\partial_t w' + \Delta w')\mathrm{d}x\mathrm{d}\tau = s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3\xi\cdot w'\mathrm{d}x\mathrm{d}\tau + \int_Q \mathbf{1}_\omega u\cdot w'\mathrm{d}x\mathrm{d}\tau$$

which is the weak transposition formulation of (4.19), and (4.22) becomes

$$\begin{aligned}\int_Q e^{-2s\alpha}|z|^2\mathrm{d}x\mathrm{d}\tau + s^{-3}\lambda^{-4}\int_0^T\int_\omega e^{-2s\alpha}\varphi^{-3}|u|^2\mathrm{d}x\mathrm{d}\tau \\ \leq Cs^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|\xi|^2\mathrm{d}x\mathrm{d}\tau.\end{aligned} \tag{4.23}$$

Let us now give an estimate of ∇z. For $\epsilon > 0$ we introduce

$$\begin{aligned}\alpha_\epsilon(t,x) &\overset{\text{def}}{=} \frac{e^{\lambda(\eta(x)+(k+1)\|\eta\|_\infty)} - e^{\lambda(k+2+\frac{1}{k})\|\eta\|_\infty}}{(\ell(t)+\epsilon)^k},\\ \varphi_\epsilon(t,x) &\overset{\text{def}}{=} \frac{e^{\lambda(\eta(x)+(k+1)\|\eta\|_\infty)}}{(\ell(t)+\epsilon)^k}.\end{aligned}$$

Thus, by multiplying the first equality in (4.19) by $s^{-2}\lambda^{-2}e^{-2s\alpha_\epsilon}\varphi_\epsilon^{-2}z$ and integrating by parts (α_ϵ, φ_ϵ do not blow up as $t \to 0$ or T) we obtain:

$$\begin{aligned}
&s^{-2}\lambda^{-2}\int_Q e^{-2s\alpha_\epsilon}\varphi_\epsilon^{-2}|\nabla z|^2 \mathrm{d}x\mathrm{d}\tau + s^{-2}\lambda^{-2}\int_\Sigma e^{-2s\widehat{\alpha}_\epsilon}\widehat{\varphi}_\epsilon^{-2}\gamma|z_n|^2\mathrm{d}\Sigma\\
&\quad = s^{-2}\lambda^{-2}\int_0^T\int_\omega e^{-2s\alpha_\epsilon}\varphi_\epsilon^{-2}u\cdot z\mathrm{d}x\mathrm{d}\tau - s^{-2}\lambda^{-2}\int_Q (z\,{}^t\nabla(e^{-2s\alpha_\epsilon}\varphi_\epsilon^{-2})) : \nabla z\mathrm{d}x\mathrm{d}\tau\\
&\qquad + \frac{1}{2}s^{-2}\lambda^{-2}\int_Q (e^{-2s\alpha_\epsilon}\varphi_\epsilon^{-2})_t|z|^2\mathrm{d}x\mathrm{d}\tau + s\lambda^2\int_Q \varphi_\epsilon\xi\cdot z\mathrm{d}x\mathrm{d}\tau.
\end{aligned}$$

Moreover, for $s \geq C(T^k + T^{k-1})$ and $C > 0$ large enough we verify that have

$$\begin{aligned}
&s^{-2}\lambda^{-2}|(e^{-2s\alpha_\epsilon}\varphi_\epsilon^{-2})_t| \leq C\lambda^{-2}e^{-2s\alpha_\epsilon},\\
&s^{-2}\lambda^{-2}|\nabla(e^{-2s\alpha_\epsilon}\varphi_\epsilon^{-2})| \leq Cs^{-1}\lambda^{-1}e^{-2s\alpha_\epsilon}\varphi_\epsilon^{-1}.
\end{aligned}$$

Then combining this with the above equality we deduce that:

$$\begin{aligned}
s^{-2}\lambda^{-2}\int_Q e^{-2s\alpha_\epsilon}\varphi_\epsilon^{-2}|\nabla z|^2\mathrm{d}x\mathrm{d}\tau &\leq C\bigg(\int_Q e^{-2s\alpha_\epsilon}|z|^2\mathrm{d}x\mathrm{d}\tau\\
&+ s^{-3}\lambda^{-4}\int_0^T\int_\omega e^{-2s\alpha_\epsilon}\varphi_\epsilon^{-3}|u|^2\mathrm{d}x\mathrm{d}\tau + s^3\lambda^4\int_Q e^{2s\alpha_\epsilon}\varphi_\epsilon^3|\xi|^2\mathrm{d}x\mathrm{d}\tau\bigg),
\end{aligned}$$

for a constant $C > 0$ independent on ϵ. Then we can pass to the limit $\epsilon \to 0$ and with (4.23) it yields (4.20).

We are now in position to deduce the estimate (4.18) for ξ by duality. By multiplying the first equation in (4.7) by z and integrate by parts we get

$$\begin{aligned}
s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|\xi|^2\mathrm{d}x\mathrm{d}\tau &= \int_Q g_1\cdot z\mathrm{d}x\mathrm{d}\tau\\
&+ \int_Q g_2\cdot \operatorname{curl} z\mathrm{d}x\mathrm{d}\tau - \int_0^T\int_\omega u\cdot\xi\mathrm{d}x\mathrm{d}\tau.
\end{aligned}\tag{4.24}$$

Then the above equality with (4.20) yields:

$$\begin{aligned}
s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|\xi|^2\mathrm{d}x\mathrm{d}\tau &\leq C\bigg(\int_Q e^{2s\alpha}|g_1|^2\mathrm{d}x\mathrm{d}\tau\\
&+ s^2\lambda^2\int_Q e^{2s\alpha}\varphi^2|g_2|^2\mathrm{d}x\mathrm{d}\tau + s^3\lambda^4\int_0^T\int_\omega e^{2s\alpha}\varphi^3|\xi|^2\mathrm{d}x\mathrm{d}\tau\bigg).
\end{aligned}\tag{4.25}$$

Finally, to get an estimate of $\nabla\xi$ we multiply by $s\lambda^2e^{2s\alpha}\varphi\xi$ the Eq. (4.7) satisfied by ξ and we integrate by parts to obtain

$$\begin{aligned}
&s\lambda^2\int_Q e^{2s\alpha}\varphi|\nabla\xi|^2\mathrm{d}x\mathrm{d}\tau + s\lambda^2\int_Q e^{2s\alpha}\varphi\gamma|\xi_n|^2\mathrm{d}x\mathrm{d}\tau =\\
&\quad + s\lambda^2\int_Q e^{2s\alpha}\varphi g_1\cdot\xi\mathrm{d}x\mathrm{d}\tau + s\lambda^2\int_Q \nabla(e^{2s\alpha}\varphi)\times\xi\cdot g_1\mathrm{d}x\mathrm{d}\tau\\
&\quad + s\lambda^2\int_Q e^{2s\alpha}\varphi g_2\cdot\operatorname{curl}\xi\mathrm{d}x\mathrm{d}\tau - \frac{1}{2}s\lambda^2\int_Q (e^{2s\alpha}\varphi)_t|\xi|^2\mathrm{d}x\mathrm{d}\tau\\
&\quad - s\lambda^2\int_Q (\xi\,{}^t\nabla(e^{2s\alpha}\varphi)) : \nabla\xi\mathrm{d}x\mathrm{d}\tau.
\end{aligned}$$

Then since for $s \geq C(T^k + T^{k-1})$ and $C > 0$ large enough we have $s\lambda^2|(e^{2s\alpha}\varphi)_t| \leq Cs^3\lambda^2e^{2s\alpha}\varphi^3$ and $s\lambda^2|\nabla(e^{2s\alpha}\varphi)| \leq Cs^2\lambda^3e^{2s\alpha}\varphi^2$, we deduce (4.18) from (4.25). □

5. Proofs of Theorem 3.3 and of Theorem 3.4

5.1. Null Controllability of the Linear System: Proof of Theorem 3.3

Assume $(g_1, g_2) \in \mathcal{G}$ and $(w_0, \theta_0) \in \mathbf{V}_0^{1/2}(\Omega) \times \widehat{\mathbf{V}}_n^{1/2}(\Omega)$ and recall first that according to Definition 6.12 of the appendix, the pair $(w, \theta) \in \mathbf{L}^2(Q) \times \mathbf{L}^2(Q)$ is a transposition solution of (1.19) if the following equality holds:

$$\int_Q (w \cdot \mathcal{L}_1^*(y, \pi, \rho) + \theta \cdot \mathcal{L}_2^*(y, \kappa, \rho, \mu))\mathrm{d}x\mathrm{d}\tau = \int_0^T \langle (g_1, g_2)|(y, \rho)\rangle_{\mathcal{V}', \mathcal{V}}\mathrm{d}\tau$$
$$+ \int_0^T \int_{\mathcal{O}} (u_1 \cdot y + u_2 \cdot P_{\mathcal{O}}\rho)\mathrm{d}x\mathrm{d}\tau + \int_\Omega (w_0 \cdot y(0) + \theta_0 \cdot \rho(0))\mathrm{d}x$$

for all $(y, \pi, \rho, \kappa, \mu) \in \mathcal{R}_0$ with $\mathcal{R}_0$ and $\mathcal{L}_i^*$, $i = 1, 2$, defined in (6.21), (6.22) in appendix and $\mathcal{V} \stackrel{\text{def}}{=} \mathbf{V}_0^1(\Omega) \times \widehat{\mathbf{V}}_n^1(\Omega)$.

Thus, we define the following symmetric bilinear form on $\mathcal{R}_0$:

$$b((y, \pi, \rho, \kappa, \mu), (y', \pi', \rho', \kappa', \mu')) = \int_Q e^{s\frac{3\widehat{\beta}}{2}}(\mathcal{L}_1^*(y, \pi, \rho) \cdot \mathcal{L}_1^*(y', \pi', \rho')\mathrm{d}x\mathrm{d}\tau$$
$$+ \int_Q e^{s\frac{3\widehat{\beta}}{2}}\mathcal{L}_2^*(y, \kappa, \rho, \mu) \cdot \mathcal{L}_2^*(y', \kappa', \rho', \mu'))\mathrm{d}x\mathrm{d}\tau$$
$$+ \int_0^T \int_{\mathcal{O}} e^{s\frac{3\widehat{\beta}}{2}}(y \cdot y' + P_{\mathcal{O}}\rho \cdot P_{\mathcal{O}}\rho')\mathrm{d}x\mathrm{d}\tau,$$

and the following linear form on $\mathcal{R}_0$:

$$l((y', \pi', \rho', \kappa', \mu')) = \int_0^T \langle (g_1, g_2)|(y', \rho')\rangle_{\mathcal{V}', \mathcal{V}}\mathrm{d}\tau$$
$$+ \int_\Omega w_0 \cdot y'(0)\mathrm{d}x + \int_\Omega \theta_0 \cdot \rho'(0)\mathrm{d}x.$$

Inequality (3.12) guarantees that the semi norm induced by $b(\cdot, \cdot)$ is in fact a norm. Then we denote by $\mathcal{R}$ the completion of $\mathcal{R}_0$ for this norm and, since $e^{-s\widehat{\beta}}(g_1, g_2) \in L^2(0, T; \mathbf{V}_0^{-1}(\Omega) \times [\mathbf{H}_n^1(\Omega)]')$ and $(w_0, \theta_0) \in \mathbf{V}_n^0(\Omega) \times \widehat{\mathbf{V}}_n^0(\Omega)$, one also has that l is continuous on $\mathcal{R}$. Then by invoking Lax-Milgram Lemma we obtain a unique $(y, \pi, \rho, \kappa, \mu) \in \mathcal{R}$ satisfying for all $(y', \pi', \rho', \kappa', \mu') \in \mathcal{R}$:

$$b((y, \pi, \rho, \kappa, \mu), (y', \pi', \rho', \kappa'.\mu')) = l((y', \pi', \rho', \kappa', \mu')). \tag{5.1}$$

Moreover, we have $\|(y, \pi, \rho, \kappa, \mu)\|_{\mathcal{R}} \leq C\|l\|_{\mathcal{R}'}$ or equivalently:

$$\int_Q e^{s\frac{3\widehat{\beta}}{2}}(|\mathcal{L}_1^*(y, \pi, \rho)|^2 + |\mathcal{L}_2^*(y, \kappa, \rho, \mu)|^2)\mathrm{d}x\mathrm{d}\tau$$
$$+ \int_0^T \int_{\mathcal{O}} e^{s\frac{3\widehat{\beta}}{2}}(|y|^2 + |P_{\mathcal{O}}\rho|^2)\mathrm{d}x\mathrm{d}\tau$$
$$\leq C\bigg(\|(g_1, g_2)\|_{\mathcal{G}}^2 + \|(w_0, \theta_0)\|_{\mathbf{V}_n^0(\Omega) \times \widehat{\mathbf{V}}_n^0(\Omega)}^2\bigg). \tag{5.2}$$

Thus, if we set $(w, \theta) = (e^{s\frac{3\widehat{\beta}}{2}}\mathcal{L}_1^*(y, \pi, \rho), e^{s\frac{3\widehat{\beta}}{2}}\mathcal{L}_2^*(y, \kappa, \rho, \mu))$ and $(u_1, u_2) = (-e^{s\frac{3\widehat{\beta}}{2}}y, -e^{s\frac{3\widehat{\beta}}{2}}P_{\mathcal{O}}\rho)$ then (5.1) implies that (w, θ) is a transposition solution of (1.19) and (5.2) yields

$$\begin{aligned}\int_Q e^{-s\frac{3\widehat{\beta}}{2}}(|w|^2 + |\theta|^2)\mathrm{d}x\mathrm{d}\tau + \int_0^T \int_{\mathcal{O}} e^{-s\frac{3\widehat{\beta}}{2}}(|u_1|^2 + |u_2|^2)\mathrm{d}x\mathrm{d}\tau \\ \leq C\bigg(\|(g_1, g_2)\|_{\mathcal{G}}^2 + \|(w_0, \theta_0)\|_{\mathbf{V}_n^0(\Omega) \times \widehat{\mathbf{V}}_n^0(\Omega)}^2\bigg).\end{aligned} \tag{5.3}$$

Finally, since $(e^{-s\frac{\widehat{\beta}}{2}}w, e^{-s\frac{\widehat{\beta}}{2}}\theta)$ satisfies (1.19) with nonhomogeneous terms

$$e^{-s\frac{\widehat{\beta}}{2}}g_1 - s\frac{\widehat{\beta}_t}{2}e^{-s\frac{\widehat{\beta}}{2}}w + e^{-s\frac{\widehat{\beta}}{2}}\mathbf{1}_{\mathcal{O}}u_1 \in L^2(0,T;\mathbf{H}^{-1/2}(\Omega)),$$
$$e^{-s\frac{\widehat{\beta}}{2}}g_2 - s\frac{\widehat{\beta}_t}{2}e^{-s\frac{\widehat{\beta}}{2}}\theta + e^{-s\frac{\widehat{\beta}}{2}}\mathbf{1}_{\mathcal{O}}P_{\mathcal{O}}u_2 \in L^2(0,T;[\mathbf{H}^{1/2}_n(\Omega)]'),$$

in transposition sense, and since by Proposition 6.13 in Appendix we have that transposition and weak solutions of (1.19) coincide, then by combining (5.3) with Theorem 6.9 in Appendix we get the estimate in Theorem 3.3.

5.2. Null Controllability of the Nonlinear System: Proof of Theorem 3.4

The first part of the theorem will follow from the Banach fixed point theorem applied to the mapping $\Psi : (\widehat{w},\widehat{\theta}) \mapsto (w,\theta)$ where (w,θ) is the solution given by Theorem 3.3 of the control problem (1.19) for $g_1 = g_1(\widehat{w},\widehat{\theta})$ and $g_2 = g_2(\widehat{w},\widehat{\theta})$ where

$$g_1(w,\theta) \stackrel{\text{def}}{=} (\theta\cdot\nabla)\theta - (w\cdot\nabla)w \quad \text{and} \quad g_2(w,\theta) \stackrel{\text{def}}{=} \operatorname{curl}(w\times\theta).$$

For that, we first need some estimates for the nonlinear terms $g_1(w,\theta)$ and $g_2(w,\theta)$. From $\operatorname{curl}(w\times\theta) = (\theta\cdot\nabla)w - (w\cdot\nabla)\theta$ and from classical estimates of Navier–Stokes type nonlinearity one has:

$$\|g_1(w,\theta)\|_{\mathbf{H}^{-1/2}(\Omega)} \le C(\|\theta\|_{\mathbf{H}^{1/2}(\Omega)}\|\theta\|_{\mathbf{H}^{3/2}(\Omega)} + \|w\|_{\mathbf{H}^{1/2}(\Omega)}\|w\|_{\mathbf{H}^{3/2}(\Omega)})$$
$$\|g_2(w,\theta)\|_{[\mathbf{H}^{1/2}_n(\Omega)]'} \le C(\|\theta\|_{\mathbf{H}^{1/2}(\Omega)}\|w\|_{\mathbf{H}^{3/2}(\Omega)} + \|w\|_{\mathbf{H}^{1/2}(\Omega)}\|\theta\|_{\mathbf{H}^{3/2}(\Omega)}).$$

Then we deduce that

$$\begin{aligned}&\|e^{-s\widehat{\beta}}(g_1(w,\theta),g_2(w,\theta))\|_{L^2(0,T;\mathbf{H}^{-1/2}(\Omega)\times[\mathbf{H}^{1/2}_n(\Omega)]')}\\ &\quad\le C(\|e^{-s\frac{\widehat{\beta}}{2}}(w,\theta)\|_{L^\infty(0,T;\mathbf{V}^{1/2}_0(\Omega)\times\widehat{\mathbf{V}}^{1/2}_n(\Omega))}\|e^{-s\frac{\widehat{\beta}}{2}}(w,\theta)\|_{L^2(0,T;\mathbf{V}^{3/2}_0(\Omega)\times\widehat{\mathbf{V}}^{3/2}_n(\Omega))},\end{aligned}$$

and the continuous embedding $\mathcal{W} \hookrightarrow L^\infty(0,T;\mathbf{V}^{1/2}_0(\Omega)\times\widehat{\mathbf{V}}^{1/2}_n(\Omega))$ yields

$$\|(g_1(w,\theta),g_2(w,\theta))\|_{\mathcal{G}} \le C\|(w,\theta)\|^2_{\mathcal{W}}. \tag{5.4}$$

Moreover, from analogous calculations we deduce that

$$\begin{aligned}&\|(g_1(w_1,\theta_1),g_2(w_1,\theta_1)) - (g_1(w_2,\theta_2),g_2(w_2,\theta_2))\|_{\mathcal{G}}\\ &\quad\le C(\|(w_1,\theta_1)\|_{\mathcal{W}} + \|(w_2,\theta_2)\|_{\mathcal{W}})\|(w_1-w_2,\theta_1-\theta_2)\|_{\mathcal{W}}.\end{aligned} \tag{5.5}$$

Thus, by combining (5.4), (5.5) and Theorem 3.3 we deduce that

$$\|\Psi(w,\theta)\|_{\mathcal{W}} \le c(\|(w,\theta)\|^2_{\mathcal{W}} + \|(w_0,\theta_0)\|_{\mathbf{V}^{1/2}_0(\Omega)\times\widehat{\mathbf{V}}^{1/2}_n(\Omega)}),$$
$$\begin{aligned}\|\Psi(w_1,\theta_1) - \Psi(w_2,\theta_2)\|_{\mathcal{W}} &\le c(\|(w_1,\theta_1)\|_{\mathcal{W}} + \|(w_2,\theta_2)\|_{\mathcal{W}})\\ &\quad\times\|(w_1-w_2,\theta_1-\theta_2)\|_{\mathcal{W}},\end{aligned}$$

for some constant $c > 0$ independent on w,θ,w_i,θ_i, $i=1,2$ and of (w_0,θ_0). Then if we set $B_0 \stackrel{\text{def}}{=} \{(w,\theta)\in\mathcal{W} \mid \|(w,\theta)\|_{\mathcal{W}} \le \delta\|(w_0,\theta_0)\|_{\mathbf{V}^{1/2}_0(\Omega)\times\widehat{\mathbf{V}}^{1/2}_n(\Omega)}\}$ one verifies that if $\|(w_0,\theta_0)\|_{\mathbf{V}^{1/2}_0(\Omega)\times\widehat{\mathbf{V}}^{1/2}_n(\Omega)} \le \varepsilon$ and if $\delta > 0$ is large enough and $\varepsilon > 0$ is small enough such that $c(\delta^2\varepsilon+1) < \delta$ and $2c\delta\varepsilon < 1$, the mapping Ψ is a contraction of B_0 into itself and (1.17)–(1.18) admits a unique solution in B_0. Then the first part of Theorem 3.4 is proved.

The second part follows from a classical argument that we recall for the sake of completeness. If $(\tilde{w},\tilde{\theta})$ is another solution of (1.17) in $C(0,T;\mathbf{H}^{1/2}(\Omega)\times\mathbf{H}^{1/2}(\Omega))\cap L^2(0,T;\mathbf{H}^{3/2}(\Omega)\times\mathbf{H}^{3/2}(\Omega))$ then

$(z, \vartheta) = (w - \tilde{w}, \theta - \tilde{\theta})$ satisfies

$$\begin{cases} z_t - \Delta z + (z \cdot \nabla)(\overline{v} + w) + ((\overline{v} + \tilde{w}) \cdot \nabla) z \\ \qquad -((\overline{B} + \theta) \cdot \nabla)\vartheta - (\vartheta \cdot \nabla)(\overline{B} + \tilde{\theta}) + \nabla \pi = 0 & \text{in } Q, \\ \vartheta_t + \operatorname{curl}(\operatorname{curl} \vartheta) - ((\overline{B} + \tilde{\theta}) \cdot \nabla) z + (z \cdot \nabla)(\overline{B} + \theta) \\ \qquad -(\vartheta \cdot \nabla)(\overline{v} + w) + ((\overline{v} + \tilde{w}) \cdot \nabla)\vartheta = 0 & \text{in } Q, \\ \operatorname{div} z = \operatorname{div} \vartheta = 0 & \text{in } Q, \\ z = 0, \quad \vartheta \cdot n = 0 & \text{on } \Sigma, \\ (\operatorname{curl} \vartheta - \overline{v} \times \vartheta) \times n = 0 & \text{on } \Sigma, \\ z(0) = 0, \quad \vartheta(0) = 0 & \text{in } \Omega. \end{cases} \tag{5.6}$$

To obtain (5.6) we have used the formula $\operatorname{curl}(a \times b) = (b \cdot \nabla) a - (a \cdot \nabla) b$ for divergence free vector fields a, b. Then by multiplying the first equality by z and the second equality by ϑ, by integrating over Ω and by making an integration by parts we obtain:

$$\frac{1}{2} \frac{d}{dt}(\|z\|^2_{\mathbf{L}^2(\Omega)} + \|\vartheta\|^2_{\mathbf{L}^2(\Omega)}) + \|\nabla z\|^2_{\mathbf{L}^2(\Omega)} + \|\operatorname{curl} \vartheta\|^2_{\mathbf{L}^2(\Omega)} = -\int_\Omega (\overline{v} \cdot \nabla)\vartheta \cdot \vartheta \mathrm{d}x$$
$$\times \int_\Omega (\vartheta \cdot \nabla)(\overline{v} + \tilde{w}) \cdot \vartheta \mathrm{d}x - \int_\Omega (z \cdot \nabla)(\overline{v} + w) \cdot z \mathrm{d}x - \int_\Omega (D^a(\overline{B} + \tilde{\theta}) z \cdot \vartheta \cdot \mathrm{d}x.$$

Thus, from Sobolev embeddings we have the estimates:

$$\begin{aligned} \left| \int_\Omega (\overline{v} \cdot \nabla)\vartheta \cdot \vartheta \mathrm{d}x \right| &\le C \|\overline{v}\|_{\mathbf{H}^2(\Omega)} \|\vartheta\|_{\mathbf{H}^1(\Omega)} \|\vartheta\|_{\mathbf{L}^2(\Omega)}, \\ \left| \int_\Omega (\vartheta \cdot \nabla)(\overline{v} + \tilde{w}) \cdot \vartheta \mathrm{d}x \right| &\le C \|\overline{v} + \tilde{w}\|_{\mathbf{H}^{3/2}(\Omega)} \|\vartheta\|^2_{\mathbf{H}^{1/2}(\Omega)}, \\ \left| \int_\Omega (z \cdot \nabla)(\overline{v} + w) \cdot z \mathrm{d}x \right| &\le C \|\overline{v} + w\|_{\mathbf{H}^{3/2}(\Omega)} \|z\|^2_{\mathbf{H}^{1/2}(\Omega)}, \\ \left| \int_\Omega D^a(\overline{B} + \tilde{\theta}) z \cdot \vartheta \mathrm{d}x \right| &\le C \|\overline{B} + \tilde{w}\|_{\mathbf{H}^{3/2}(\Omega)} \|z\|_{\mathbf{H}^{1/2}(\Omega)} \|\vartheta\|_{\mathbf{H}^{1/2}(\Omega)}, \end{aligned}$$

and the use of interpolation inequality $\| \cdot \|_{\mathbf{H}^{1/2}(\Omega)} \le C \| \cdot \|^{1/2}_{\mathbf{H}^1(\Omega)} \| \cdot \|^{1/2}_{\mathbf{L}^2(\Omega)}$, Cauchy Schwartz inequality, (2.2) and Poincaré inequality, yields

$$\begin{aligned} &\frac{d}{dt}(\|z\|^2_{\mathbf{L}^2(\Omega)} + \|\vartheta\|^2_{\mathbf{L}^2(\Omega)}) + \|\nabla z\|^2_{\mathbf{L}^2(\Omega)} + \|\operatorname{curl} \vartheta\|^2_{\mathbf{L}^2(\Omega)} \\ &\quad \le C(\|\overline{v}\|^2_{\mathbf{H}^2(\Omega)} + \|\overline{B}\|^2_{\mathbf{H}^2(\Omega)} + \|w\|^2_{\mathbf{H}^{3/2}(\Omega)} + \|\tilde{w}\|^2_{\mathbf{H}^{3/2}(\Omega)}) \\ &\qquad \times (\|z\|^2_{\mathbf{L}^2(\Omega)} + \|\vartheta\|^2_{\mathbf{L}^2(\Omega)}). \end{aligned}$$

Finally, by Gronwall's Lemma we get that z and ϑ are identically zero.

6. Appendix

6.1. Regularity Results for Stokes Problem with Non Standard Conditions

The present section is dedicated to regularity results for the following Stokes system with a nonhomogeneous non standard boundary condition:

$$\begin{cases} \theta_t + \operatorname{curl}(\operatorname{curl}\theta) + \nabla\chi + \displaystyle\sum_{i=1}^{N}\beta_i \mathrm{g}_i = g & \text{in } Q, \\ \operatorname{div}\theta = 0 & \text{in } Q, \\ \forall i = 1,\dots,N \quad \displaystyle\int_\Omega \theta\cdot \mathrm{g}_i \mathrm{d}x = 0 & \text{in } (0,T), \\ \theta\cdot n = 0, \quad \operatorname{curl}\theta\times n = b\times n & \text{on } \Sigma, \\ \theta(0) = \theta_0 & \text{in } \Omega. \end{cases} \tag{6.1}$$

Here, for $g \in \mathbf{L}^2(Q)$ and $\theta_0 \in \widehat{\mathbf{V}}^1_n(\Omega)$ we aim at providing regularity results for a boundary datum $b \in \mathbf{H}^{\frac{r}{2},\frac{r}{4}}(\Sigma)$ for $r \in [0,1]$.

But before to deal with system (6.1), we need preliminary lemmas.

Lemma 6.1. *For $f \in \mathbf{L}^2(\Omega)$ satisfying $\int_\Omega f\cdot \mathrm{g}_i \mathrm{d}x = 0$, $i = 1,\dots,N$ and $h \in H^1_0(\Omega)$ satisfying $\int_\Omega h\mathrm{d}x = 0$ the following Stokes system with non standard boundary conditions*

$$\begin{cases} \operatorname{curl}(\operatorname{curl}\theta) + \nabla\chi = f & \textit{in } \Omega, \\ \operatorname{div}\theta = h & \textit{in } \Omega, \\ \theta\cdot n = 0 & \textit{on } \partial\Omega, \\ \operatorname{curl}\theta\times n = 0 & \textit{on } \partial\Omega, \end{cases} \tag{6.2}$$

admits a unique solution $(\theta,\chi) \in \mathbf{H}^2(\Omega)\times H^1(\Omega)$ satisfying $\int_\Omega \theta\cdot \mathrm{g}_i \mathrm{d}x = 0$ for all $i = 1,\dots,N$, and $\int_\Omega \chi \mathrm{d}x = 0$. Moreover, the following estimate holds

$$\|\theta\|_{\mathbf{H}^2(\Omega)} + \|\chi\|_{H^1(\Omega)} \le C(\|f\|_{\mathbf{L}^2(\Omega)} + \|h\|_{H^1(\Omega)})$$

for a constant $C > 0$ independent on θ, χ, f, h.

Proof. Let us first note that we can reduce the problem to the case $h = 0$ with a lifting procedure. Indeed, since $h \in H^1_0(\Omega)$ and $\int_\Omega h\mathrm{d}x = 0$ there exists $\theta_h \in \mathbf{H}^2_0(\Omega)$ such that $\operatorname{div}\theta_h = h$ in Ω and $\|\theta_h\|_{\mathbf{H}^2(\Omega)} \le C\|h\|_{H^1(\Omega)}$ for some $C > 0$ independent on h and θ_h, see [2, Cor. 3.1]. Then θ is a solution of (6.2) if and only if $\theta - \theta_h$ satisfies (6.2) with $h = 0$ and with a nonhomogeneous body force term $f - \operatorname{curl}(\operatorname{curl}\theta_h)$ which depends continuously on $(f,h) \in \mathbf{L}^2(\Omega)\times H^1(\Omega)$.

It remains to consider the system (6.2) with an homogeneous divergence condition, namely:

$$\begin{cases} \operatorname{curl}(\operatorname{curl}\theta) + \nabla\chi = f & \text{in } \Omega, \\ \operatorname{div}\theta = 0 & \text{in } \Omega, \\ \theta\cdot n = 0 & \text{on } \partial\Omega, \\ \operatorname{curl}\theta\times n = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.3}$$

By multiplying the first equation of (6.3) by a test function $\rho \in \widehat{\mathbf{V}}^1_n(\Omega)$ we obtain the following weak formulation of (6.3):

$$\forall\rho \in \widehat{\mathbf{V}}^1_n(\Omega) \quad \int_\Omega \operatorname{curl}\theta\cdot\operatorname{curl}\rho \mathrm{d}x = \int_\Omega f\cdot\rho \mathrm{d}x. \tag{6.4}$$

By (2.2) we have that the bilinear form defined at the left of the above equality is coercive on $\widehat{\mathbf{V}}^1_n(\Omega)$ and the Lax-Milgram Lemma yields the existence and the uniqueness of $\theta \in \widehat{\mathbf{V}}^1_n(\Omega)$ solution to the variational problem (6.4). Moreover, since $\operatorname{curl}\mathrm{g}_i = 0$ in Ω and $\int_\Omega f\cdot \mathrm{g}_i \mathrm{d}x = 0$ for $i = 1,\dots,N$ the formulation (6.4) is also true for test functions ρ in $\mathbf{V}^1_n(\Omega)$. We denote by $P : \mathbf{L}^2(\Omega) \to \mathbf{V}^0_n(\Omega)$ the Helmholtz orthogonal projection operator related to Ω. From the underlying related Neumann problem (see [25, Chap.III, Lem 1.2]) we can verify that P is also continuous from $\mathbf{H}^1(\Omega)$ onto $\mathbf{V}^1_n(\Omega)$. Then for all $\rho \in \mathbf{H}^1(\Omega)$ we have $P\rho \in \mathbf{V}^1_n(\Omega)$ and

$$\forall \rho \in \mathbf{H}^1(\Omega) \quad \int_\Omega \operatorname{curl}\theta \cdot \operatorname{curl} P\rho \,\mathrm{d}x = \int_\Omega f \cdot P\rho \,\mathrm{d}x, \tag{6.5}$$

and since $\operatorname{curl} P\rho = \operatorname{curl}\rho$ in Ω we deduce that

$$\forall \rho \in \mathbf{H}^1(\Omega) \quad \int_\Omega \operatorname{curl}\theta \cdot \operatorname{curl}\rho \,\mathrm{d}x = \int_\Omega (f - \nabla\chi) \cdot \rho \,\mathrm{d}x, \tag{6.6}$$

where $\chi \in H^1(\Omega)$ is given by $\nabla\chi = (I - P)f$. It follows that $\operatorname{curl}(\operatorname{curl}\theta) = f - \nabla\chi \in \mathbf{L}^2(\Omega)$ which allows to define $\operatorname{curl}\theta \times n$ in $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ from the following formula for all $\rho \in \mathbf{H}^1(\Omega)$:

$$\langle(\operatorname{curl}\theta \times n), \rho\rangle_{\mathbf{H}^{-\frac{1}{2}}(\partial\Omega), \mathbf{H}^{\frac{1}{2}}(\partial\Omega)} = \int_\Omega \operatorname{curl}\theta \cdot \operatorname{curl}\rho \,\mathrm{d}x - \int_\Omega \operatorname{curl}(\operatorname{curl}\theta) \cdot \rho \,\mathrm{d}x.$$

Then from (6.6) we deduce that $\operatorname{curl}\theta \times n = 0$ on $\partial\Omega$. Finally, since $z \stackrel{\text{def}}{=} \operatorname{curl}\theta$ satisfies $\operatorname{curl} z \in \mathbf{L}^2(\Omega)$, $\operatorname{div} z = 0$ in Ω and $z \times n = 0$ on $\partial\Omega$ we first get $\operatorname{curl}\theta \in \mathbf{H}^1(\Omega)$ from the first statement of [1, Cor. 2.15]. Thus, from $\operatorname{curl}\theta \in \mathbf{H}^1(\Omega)$, $\operatorname{div}\theta = 0$ in Ω and $\theta \cdot n = 0$ on $\partial\Omega$ which is of class $C^{2,1}$, we finally get $\theta \in \mathbf{H}^2(\Omega)$ from the second statement of [1, Cor. 2.15]. □

The above result permits to deduce the following lifting Lemma.

Lemma 6.2. *For $b \in \mathbf{H}^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ there exists $\theta_b \in \mathbf{H}^{2,1}(Q)$ satisfying*

$$\theta_b(0) = 0 \;\; in \;\; \Omega, \quad \operatorname{div}\theta_b = 0 \;\; in \;\; Q, \tag{6.7}$$

$$\operatorname{curl}\theta_b \times n = b \times n \quad and \quad \theta_b \cdot n = 0 \;\; on \;\; \Sigma. \tag{6.8}$$

Moreover, the following estimate holds

$$\|\theta_b\|_{\mathbf{H}^{2,1}(Q)} \le c\|b\|_{\mathbf{H}^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \tag{6.9}$$

for some constant $c > 0$ independent on b.

Proof. We first invoke [27, Thm 7.2] to obtain the existence of $\vartheta_b \in \mathbf{H}^{2,1}(Q)$ depending continuously on $b \in \mathbf{H}^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ and such that $\vartheta_b(0) = 0$ in Ω, $\vartheta_b = 0$ on Σ and $\frac{\partial\vartheta_b}{\partial n} = b \times n$ on Σ. Note that since $\vartheta_b = 0$ on Σ implies $\operatorname{curl}\vartheta_b \times n = \frac{\partial\vartheta_b}{\partial n}$ on Σ we deduce that $\operatorname{curl}\vartheta_b \times n = b \times n$ on Σ. Thus, we set $\theta_b = \vartheta_b - L\vartheta_b$ where the linear mapping L is defined by $L\vartheta = \theta \in \widehat{\mathbf{V}}^2_n(\Omega)$ where θ is the solution given by Lemma 6.1 of the Stokes problem:

$$\begin{cases} \operatorname{curl}(\operatorname{curl}\theta) + \nabla\chi = 0 & \text{in } \Omega, \\ \operatorname{div}\theta = \operatorname{div}\vartheta & \text{in } \Omega, \\ \theta \cdot n = 0 & \text{on } \partial\Omega, \\ \operatorname{curl}\theta \times n = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.10}$$

It is clear that such a θ_b will satisfies (6.7) and (6.8). According to Lemma 6.1, L is continuous from $\{\vartheta \in \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega) \mid \operatorname{div}\vartheta = 0 \text{ on } \partial\Omega\}$ into $\mathbf{H}^2(\Omega)$. Then since we have $\vartheta_b(t) = 0$ on $\partial\Omega$ and $\operatorname{div}\vartheta_b(t) = \frac{\partial\vartheta_b(t)}{\partial n} \cdot n = 0$ on $\partial\Omega$, we deduce that $L\vartheta_b(t)$ is well defined and that $\|(I - L)\vartheta_b\|_{L^2(0,T;\mathbf{H}^2(\Omega))} \le C\|\vartheta_b\|_{L^2(0,T;\mathbf{H}^2(\Omega))}$. Moreover, with a transposition argument we can also prove that L is continuous for the norm of $\mathbf{L}^2(\Omega)$: if we multiply the first equation of (6.10) by ρ solution of

$$\begin{cases} \operatorname{curl}(\operatorname{curl}\rho) + \nabla\kappa = \theta & \text{in } \Omega, \\ \operatorname{div}\rho = 0 & \text{in } \Omega, \\ \rho \cdot n = 0 & \text{on } \partial\Omega, \\ \operatorname{curl}\rho \times n = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.11}$$

an integration by parts gives $\int_\Omega |\theta|^2 \mathrm{d}x = \int_\Omega \vartheta \cdot \nabla\kappa \mathrm{d}x$. Thus, by using the bound $\|\nabla\kappa\|_{\mathbf{L}^2(\Omega)} \le C\|\theta\|_{\mathbf{L}^2(\Omega)}$ given by Lemma 6.1 we get $\|L\vartheta\|_{\mathbf{L}^2(\Omega)} \le C\|\vartheta\|_{\mathbf{L}^2(\Omega)}$ and then $\|(I-L)\vartheta_b\|_{H^1(0,T;\mathbf{L}^2(\Omega))} \le C\|\vartheta_b\|_{H^1(0,T;\mathbf{L}^2(\Omega))}$. Finally, we have $\|\theta_b\|_{\mathbf{H}^{2,1}(Q)} \le C\|\vartheta_b\|_{\mathbf{H}^{2,1}(Q)}$ and the conclusion follows from the continuous dependence of θ_b with respect to b. $\square$

Finally, we are now in position to turn back to system 6.1.

Definition 6.3. For $\theta_0 \in \widehat{\mathbf{V}}^1_n(\Omega)$, $g \in \mathbf{L}^2(Q)$ and $b \in \mathbf{H}^{\frac{r}{2},\frac{r}{4}}(\Sigma)$ with $r \in [0,1]$, the function $\theta \in W(0,T;\widehat{\mathbf{V}}^1_n(\Omega),[\widehat{\mathbf{V}}^1_n(\Omega)]')$ is said to be a weak solution of (6.1) if and only if $\theta(0) = \theta_0$ and if for all $\rho \in \widehat{\mathbf{V}}^1_n(\Omega)$ and $t \in (0,T)$ we have:

$$\frac{d}{dt}\int_\Omega \theta(t)\cdot\rho \mathrm{d}x + \int_\Omega \operatorname{curl}\theta(t)\cdot\operatorname{curl}\rho \mathrm{d}x = \int_\Omega g(t)\cdot\rho \mathrm{d}x + \int_{\partial\Omega}(b(t)\times n)\cdot\rho \mathrm{d}\Gamma.$$

Remark 6.4. As usual, the above equality is obtained from (6.1) by (formally) multiplying the first equality by ρ and integrating by parts. Conversely, if θ is a regular weak solution, for instance $\theta \in W(0,T;\widehat{\mathbf{V}}^2_n(\Omega),\widehat{\mathbf{V}}^0_n(\Omega))$, then an integration by parts and the use of (2.3) permits to recover the formulation (6.1) in a classical sense for some $\chi \in L^2(0,T;H^1(\Omega))$ and $\beta_i \in L^2(0,T)$.

Theorem 6.5. *Assume $r \in [0,1]$ and $\epsilon \in (0,\frac{1}{2})$. For all $\theta_0 \in \widehat{\mathbf{V}}^1_n(\Omega)$, $g \in \mathbf{L}^2(Q)$ and $b \in \mathbf{H}^{\frac{r}{2},\frac{r}{4}}(\Sigma)$ the system (6.1) admits a unique weak solution θ. Moreover, $\theta \in \mathbf{H}^{\frac{3}{2}-\epsilon+(\frac{1}{2}+\epsilon)r,\frac{3}{4}-\frac{\epsilon}{2}+(\frac{1}{4}+\frac{\epsilon}{2})r}(Q)$ and satisfies:*

$$\|\theta\|_{\mathbf{H}^{\frac{3}{2}-\epsilon+(\frac{1}{2}+\epsilon)r,\frac{3}{4}-\frac{\epsilon}{2}+(\frac{1}{4}+\frac{\epsilon}{2})r}(Q)} \le c(\|b\|_{\mathbf{H}^{\frac{r}{2},\frac{r}{4}}(\Sigma)} + \|g\|_{\mathbf{L}^2(Q)} + \|\theta_0\|_{\mathbf{V}^1_n(\Omega)}), \tag{6.12}$$

for some constant $c > 0$ independent on θ, b, θ_0, g.

Proof. The weak formulation in Definition 6.3 writes

$$\theta' + \widehat{A}\theta = \widehat{P}g + Tb,$$

where $\widehat{A}$ is the linear operator defined from the bilinear form $(\theta,\rho) \in \widehat{\mathbf{V}}^1_n(\Omega)\times\widehat{\mathbf{V}}^1_n(\Omega) \mapsto \int_\Omega \operatorname{curl}\theta\cdot\operatorname{curl}\rho \mathrm{d}x$, where $\widehat{P}$ is the orthogonal projector from $\mathbf{L}^2(\Omega)$ onto $\widehat{\mathbf{V}}^0_n(\Omega)$ and where $Tb \in [\widehat{\mathbf{V}}^1_n(\Omega)]'$ is defined by $\langle Tb|\rho\rangle = \int_{\partial\Omega} b\times n\cdot\rho \mathrm{d}\Gamma$. Classical arguments guarantee that $\widehat{A}$ generates an analytic semigroup on $\mathbf{V}^0_n(\Omega)$ (see [9, Thm. 2.12, Part II, Chap 1]) and from Lemma 6.1 we deduce that $D(\widehat{A}) = \{\theta \in \mathbf{V}^2_n(\Omega) \mid \operatorname{curl}\theta\times n = 0 \text{ on } \partial\Omega\}$. Moreover, since we easily verify that $Tb \in [\widehat{\mathbf{V}}_n^{\frac{1}{2}+\epsilon}(\Omega)]'$, maximal regularity results with an interpolation argument give $\theta \in W(0,T;\widehat{\mathbf{V}}_n^{\frac{3}{2}-\epsilon}(\Omega),[\widehat{\mathbf{V}}_n^{\frac{1}{2}+\epsilon}(\Omega)]')$. Then (6.12) for $r=0$ follows because this last space is continuously embedded in $\mathbf{H}^{\frac{3}{2}-\epsilon,\frac{3}{4}-\frac{\epsilon}{2}}(Q)$.

It remains to prove (6.12) for $r=1$, the case $r\in(0,1)$ will then follow by interpolation. According to Lemma 6.2 there exists $\theta_b \in \mathbf{H}^{2,1}(Q)$ satisfying (6.7), (6.8) and depending continuously on $b \in \mathbf{H}^{\frac{1}{2},\frac{1}{4}}(\Sigma)$. Then $\vartheta = \theta - \theta_b$ satisfies

$$\vartheta' + \widehat{A}\vartheta = \widehat{P}\tilde{g}, \quad \vartheta(0) = \theta_0$$

for $\tilde{g} = g - (\theta_b)_t - \operatorname{curl}(\operatorname{curl}\theta_b) \in \mathbf{L}^2(Q)$, and (6.12) for $r=1$ follows from classical maximal regularity results for analytic semigroup and from the continuity of $b \in \mathbf{H}^{\frac{1}{2},\frac{1}{4}}(\Sigma) \mapsto \theta_b \in \mathbf{H}^{2,1}(Q)$. $\square$

6.2. Regularity Results for Linear MHD Equations

The present subsection deals with the well-posedness of the following non autonomous linear magneto-hydrodynamic equations:

$$\begin{cases} w_t - \Delta w + (w\cdot\nabla)\overline{v} + (\overline{v}\cdot\nabla)w - (\overline{B}\cdot\nabla)\theta & \\ \qquad\qquad -(\theta\cdot\nabla)\overline{B} + \nabla q = g_1 & \text{in } Q,\\ \theta_t + \operatorname{curl}(\operatorname{curl}\theta) - \operatorname{curl}(w\times\overline{B} + \overline{v}\times\theta) & \\ \qquad\qquad +\nabla\chi + \sum_{i=1}^{N}\beta_i \mathrm{g}_i = g_2 & \text{in } Q,\\ \operatorname{div} w = \operatorname{div}\theta = 0 & \text{in } Q,\\ \forall i = 1,\dots,N \quad \int_\Omega \theta\cdot \mathrm{g}_i \mathrm{d}x = 0 & \text{in } (0,T),\\ w = 0, \quad \theta\cdot n = 0 & \text{on } \Sigma,\\ (\operatorname{curl}\theta - \overline{v}\times\theta)\times n = 0 & \text{on } \Sigma,\\ w(0) = w_0, \quad \theta(0) = \theta_0 & \text{in } \Omega. \end{cases} \tag{6.13}$$

For the following we assume $(g_1, g_2) \in L^2(0,T;\mathbf{H}^{-1}(\Omega)) \times [\mathbf{H}^1_n(\Omega))]')$ and $(w_0,\theta_0) \in \mathbf{V}^0_n(\Omega)\times\widehat{\mathbf{V}}^0_n(\Omega)$. In (6.13) the pressure functions q, χ and the real values β_i, $i = 1,\dots,N$ play the role of the Lagrange multiplyers related to constraints $\operatorname{div} w = \operatorname{div}\theta = 0$ in Q and $\int_\Omega \theta\cdot\mathrm{g}_i\mathrm{d}x = 0$, $i = 1,\dots,N$ in $(0,T)$.

Remark 6.6. If g_2 is of the form $\operatorname{curl}(w\times\theta) + \mathbf{1}_{\mathcal{O}}P_{\mathcal{O}}u_2$ then we have $\nabla\chi = 0$ in Q and $\beta_i = 0$, $i = 1,\dots,N$, and the induction equation in (1.19) is recovered. This can be obtained by successively multiplying the equation by $\nabla\chi$ and g_i, integrating by parts and using (1.11), $\mathrm{g}_i \in \mathbf{V}^0_n(\Omega)$ and $\operatorname{curl}\mathrm{g}_i = 0$ in Ω.

We first introduce the notion of weak solution of (6.13). For that we set

$$\mathcal{H} \stackrel{\text{def}}{=} \mathbf{V}^0_n(\Omega)\times\widehat{\mathbf{V}}^0_n(\Omega) \quad \text{and} \quad \mathcal{V} \stackrel{\text{def}}{=} \mathbf{V}^1_0(\Omega)\times\widehat{\mathbf{V}}^1_n(\Omega)$$

and we define the family of linear operators $A(t) : \mathcal{V} \to \mathcal{V}'$, $t \in [0,T]$ by

$$\begin{aligned} \langle A(t)(w,\theta)\ |(y,\rho)\rangle_{\mathcal{V},\mathcal{V}'} \stackrel{\text{def}}{=} & \int_\Omega (\nabla w : \nabla y + \operatorname{curl}\theta\cdot\operatorname{curl}\rho)\mathrm{d}x \\ & + \int_\Omega ((w\cdot\nabla)\overline{v}(t) + (\overline{v}(t)\cdot\nabla)w - (\overline{B}(t)\cdot\nabla)\theta - (\theta\cdot\nabla)\overline{B}(t))\cdot y\mathrm{d}x \\ & - \int_\Omega (w\times\overline{B}(t) + \overline{v}(t)\times\theta)\cdot\operatorname{curl}\rho\mathrm{d}x. \end{aligned}$$

Definition 6.7. Assume $(g_1, g_2) \in L^2(0,T;\mathbf{H}^{-1}(\Omega)) \times [\mathbf{H}^1_n(\Omega))]')$ and $(w_0,\theta_0)\in\mathcal{H}$. We say that $(w,\theta) \in W(0,T;\mathcal{V},\mathcal{V}')$ is a weak solution of (6.13) if it satisfies for all $(y,\rho)\in\mathcal{V}$ and $t\in(0,T)$:

$$\begin{aligned} & \frac{d}{dt}((w(t),\theta(t))|(y,\rho))_{\mathcal{H}} \\ & \quad + \langle A(t)(w(t),\theta(t)) \mid (y,\rho)\rangle_{\mathcal{V},\mathcal{V}'} = \langle (g_1(t),g_2(t)) \mid (y,\rho)\rangle_{\mathcal{V},\mathcal{V}'} \\ & \quad (w(0),\theta(0)) = (w_0,\theta_0). \end{aligned} \tag{6.14}$$

Remark 6.8. Such a notion of weak solution is coherent because if (g_1, g_2) and (w,θ) are regular, for instance $(g_1,g_2) \in \mathbf{L}^2(Q)\times\mathbf{L}^2(Q)$ and $(w,\theta)\in W(0,T;\mathbf{V}^2_n(\Omega)\times\widehat{\mathbf{V}}^2_n(\Omega),\mathbf{V}^0_n(\Omega)\times\widehat{\mathbf{V}}^0_n(\Omega))$, then (w,θ) satisfies (6.13) in a classical sense. This can be deduced from an integration by parts and by recalling (2.1) and (2.3) to obtain the existence of $q\in L^2(0,T;H^1(\Omega))$, $\chi\in L^2(0,T;H^1(\Omega))$ and $\beta_i\in L^2(0,T)$, $i = 1,\dots,N$, such that (6.13) is satisfied.

We have the following existence theorem for weak solutions.

Theorem 6.9. *Assume* $r \in [0,1]$, $(w_0, \theta_0) \in \mathbf{V}_0^r(\Omega) \times \widehat{\mathbf{V}}_n^r(\Omega)$ *and* $(g_1, g_2) \in L^2(0,T; \mathbf{H}^{r-1}(\Omega) \times [\mathbf{H}_n^{1-r}(\Omega)]')$. *Then the system* (6.13) *admits a unique weak solution* (w, θ). *Moreover,* $(w,\theta) \in W(0,T; \mathbf{V}_0^{r+1}(\Omega) \times \widehat{\mathbf{V}}_n^{r+1}(\Omega), \mathbf{V}^{r-1}(\Omega) \times [\widehat{\mathbf{V}}_n^{1-r}(\Omega)]')$ *and satisfies:*

$$\begin{aligned} &\|(w,\theta)\|_{W(0,T;\mathbf{V}_0^{r+1}(\Omega)\times\widehat{\mathbf{V}}_n^{r+1}(\Omega),\mathbf{V}^{r-1}(\Omega)\times[\widehat{\mathbf{V}}_n^{1-r}(\Omega)]')} \\ &\quad \le C\|(g_1,g_2)\|_{L^2(0,T;\mathbf{H}^{r-1}(\Omega)\times[\mathbf{H}_n^{1-r}(\Omega)]')} + \|(w_0,\theta_0)\|_{\mathbf{V}_0^r(\Omega)\times\widehat{\mathbf{V}}_n^r(\Omega))}, \end{aligned} \tag{6.15}$$

for a constant $C > 0$ *independent on* w, θ *and on* w_0, θ_0, g_1, g_2.

Proof. From (1.8) and standard Sobolev embeddings we get that for all $(w,\theta) \in \mathcal{V}$ and $(y,\rho) \in \mathcal{V}$ the mapping $t \mapsto \langle (A(t))(w,\theta) \mid (y,\rho)\rangle_{\mathcal{V},\mathcal{V}'}$ is measurable on $(0,T)$ and that the following continuity and coercivity inequalities hold for all $t \in (0,T)$:

$$\begin{aligned} \langle A(t)(w,\theta) \mid (y,\rho)\rangle_{\mathcal{V},\mathcal{V}'} &\le c_1\|(w,\theta)\|_{\mathcal{V}}\|(y,\rho)\|_{\mathcal{V}}, \\ \langle A(t)(w,\theta) \mid (w,\theta)\rangle_{\mathcal{V},\mathcal{V}'} + c_2\|(w,\theta)\|_{\mathcal{H}}^2 &\ge c_3\|(w,\theta)\|_{\mathcal{V}}^2, \end{aligned}$$

for some $c_1 > 0$, $c_2 >$ and $c_3 > 0$ independent on t and (w,θ,y,ρ). Then according to [9, Part II, Chap. 2, Thm 1.2], for $(g_1,g_2) \in L^2(0,T;\mathbf{H}^{-1}(\Omega)\times[\mathbf{H}_n^1(\Omega)]')$ and $(w_0,\theta_0) \in \mathcal{H}$ the equations (6.14) admits a unique solution satisfying (6.15) for $r = 0$.

It remains to prove (6.15) for $r = 1$. Then the conclusion will follow by interpolation. For the following we suppose that $(g_1, g_2) \in \mathbf{L}^2(Q) \times \mathbf{L}^2(Q)$ and $(w_0,\theta_0) \in \mathbf{V}_n^1(\Omega) \times \widehat{\mathbf{V}}_n^1(\Omega)$. From (6.14) we have that w is a weak solution of the following Stokes system:

$$\begin{cases} w_t - \Delta w + \nabla q = \tilde{g}_1 & \text{in } Q, \\ \operatorname{div} w = 0 & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(0) = w_0 & \text{in } \Omega, \end{cases} \tag{6.16}$$

with $\tilde{g}_1 \overset{\text{def}}{=} g_1 - (w\cdot\nabla)\overline{v} - (\overline{v}\cdot\nabla)w + (\overline{B}\cdot\nabla)\theta + (\theta\cdot\nabla)\overline{B} \in \mathbf{L}^2(Q)$, and we have that θ is a weak solution of the following Stokes system with non standard boundary conditions:

$$\begin{cases} \theta_t + \operatorname{curl}(\operatorname{curl}\theta) + \nabla\chi + \displaystyle\sum_{i=1}^N \beta_i \mathrm{g}_i = \tilde{g}_2 & \text{in } Q, \\ \operatorname{div}\theta = 0 & \text{in } Q, \\ \forall i = 1,\dots,N \quad \displaystyle\int_\Omega \theta\cdot \mathrm{g}_i \mathrm{d}x = 0 & \text{in } (0,T), \\ \theta\cdot n = 0, \quad \operatorname{curl}\theta \times n = b \times n & \text{on } \Sigma, \\ \theta(0) = \theta_0 & \text{in } \Omega, \end{cases} \tag{6.17}$$

where $\tilde{g}_2 = g_2 + \operatorname{curl}(w \times \overline{B} + \overline{v}\times\theta) \in \mathbf{L}^2(Q)$ and $b = \overline{v}\times\theta$. Since $\tilde{g}_1 \in \mathbf{L}^2(Q)$ classical regularity results for the Stokes problem with homogeneous boundary conditions yield $w \in W(0,T;\mathbf{V}_0^2(\Omega)\times\mathbf{V}_n^0(\Omega))$. It remains to prove that $\theta \in W(0,T;\widehat{\mathbf{V}}_n^2(\Omega),\widehat{\mathbf{V}}_n^0(\Omega))$ and for that we need to combine regularity results for system (6.17) of Theorem 6.5 with a bootstrap argument.

First, note that (1.8) implies in particular that

$$\overline{v}_{|\Sigma} \in L^\infty(0,T;\mathbf{H}^{\frac{1}{2}}(\partial\Omega)) \cap H^{\frac{1}{2}}(0,T;\mathbf{H}^{\frac{1}{2}}(\partial\Omega)). \tag{6.18}$$

Then since θ given by (6.14) belongs to $W(0,T;\mathbf{V}_n^1(\Omega),\widehat{[\mathbf{V}_n^1(\Omega)]'})$ we have $\theta \in \mathbf{H}^{1,\frac{1}{2}}(Q)$ and $\theta_{|\Sigma} \in \mathbf{H}^{\frac{1}{2},\frac{1}{4}}(\Sigma)$, and with (6.18) classical Sobolev embeddings guarantee that $\overline{v}\times\theta_{|\Sigma} \in \mathbf{L}^2(\Sigma)$. Thus, from (6.12) we get that $\theta \in \mathbf{H}^{\frac{3}{2}-\epsilon,\frac{3}{4}-\frac{\epsilon}{2}}(Q)$ and then $\theta_{|\Sigma} \in \mathbf{H}^{1-\epsilon,\frac{1}{2}-\frac{\epsilon}{2}}(\Sigma)$ for $\epsilon \in (0,\frac{1}{4})$. Then with (6.18) we get $\overline{v} \times \theta_{|\Sigma} \in \mathbf{H}^{\frac{1}{2}-2\epsilon,\frac{1}{4}-\epsilon}(\Sigma)$ and (6.12) yields $\theta \in \mathbf{H}^{2-2\epsilon,1-\epsilon}(Q)$. It follows that $\theta_{|\Sigma} \in \mathbf{H}^{\frac{3}{2}-2\epsilon,\frac{3}{4}-\epsilon}(\Sigma)$ and using again (6.18) we deduce that $\overline{v} \times \theta_{|\Sigma} \in \mathbf{H}^{\frac{1}{2},\frac{1}{4}}(\Sigma)$. Finally, from (6.12) we get $\theta \in \mathbf{H}^{2,1}(Q)$ and then (6.15) holds for $r = 1$. This ends the proof. $\square$

Let us now consider the well-posedness of the adjoint system:

$$\begin{cases} -y_t - \Delta y - (D^s y)\overline{v} + (D^a \rho)\overline{B} + \nabla \pi = f_1 & \text{in } Q, \\ -\rho_t - \Delta \rho + (D^s y)\overline{B} - (D^a \rho)\overline{v} + \nabla \kappa + \displaystyle\sum_{i=1}^{N} \mu_i \mathrm{g}_i = f_2 & \text{in } Q, \\ \operatorname{div} y = \operatorname{div} \rho = 0 & \text{in } Q, \\ \forall i = 1, \ldots, N \displaystyle\int_\Omega \rho \cdot \mathrm{g}_i \mathrm{d}x = 0 & \text{in } (0,T), \\ y = 0, \quad \rho \cdot n = 0, \quad (\operatorname{curl} \rho) \times n = 0 & \text{on } \Sigma. \\ y(T) = y_T \quad \rho(T) = \rho_T & \text{in } \Omega. \end{cases} \tag{6.19}$$

Similarly as for system (6.13) we define weak solutions of (6.19) as follows.

Definition 6.10. Assume that $(f_1, f_2) \in L^2(0,T; \mathbf{H}^{-1}(\Omega)) \times [\mathbf{H}^1_n(\Omega))]')$ and that $(y_T, \rho_T) \in \mathcal{H}$. We say that $(y, \rho) \in W(0,T; \mathcal{V}, \mathcal{V}')$ is a weak solution of (6.13) if it satisfies for all $(w, \theta) \in \mathcal{V}$ and $t \in (0,T)$:

$$\begin{aligned} -\frac{d}{dt}&((w,\theta)|(y(t),\rho(t)))_{\mathcal{H}} \\ &+ \langle A(t)(w,\theta) \mid (y(t),\rho(t))\rangle_{\mathcal{V},\mathcal{V}'} = \langle (f_1(t), f_2(t)) \mid (w,\theta)\rangle_{\mathcal{V},\mathcal{V}'} \\ &(y(T),\rho(T)) = (y_T, \rho_T). \end{aligned}$$

We have the following regularity theorem for weak solutions of (6.13).

Theorem 6.11. *Assume $(y_T, \rho_T) \in \mathbf{V}^1_0(\Omega) \times \widehat{\mathbf{V}}^1_n(\Omega)$ and $(f_1, f_2) \in \mathbf{L}^2(Q) \times \mathbf{L}^2(Q)$. Then system (6.19) admits a unique weak solution (y, ρ). Moreover, (y, ρ) belongs to $W(0,T; \mathbf{V}^2_0(\Omega) \times \widehat{\mathbf{V}}^2_n(\Omega), \mathbf{V}^0_n(\Omega) \times \widehat{\mathbf{V}}^0_n(\Omega))$ and satisfies:*

$$\begin{aligned} &\|(y,\rho)\|_{W(0,T;\mathbf{V}^2_0(\Omega)\times\widehat{\mathbf{V}}^2_n(\Omega),\mathbf{V}^0_n(\Omega)\times\widehat{\mathbf{V}}^0_n(\Omega))} \\ &\quad \le C\left(\|(f_1,f_2)\|_{\mathbf{L}^2(Q)\times\mathbf{L}^2(Q)} + \|(y_T,\rho_T)\|_{\mathbf{V}^1_0(\Omega)\times\widehat{\mathbf{V}}^1_n(\Omega)}\right) \end{aligned} \tag{6.20}$$

for a constant $C > 0$ independent on y, ρ and on y_T, ρ_T, f_1, f_2.

Since the proof of Theorem 6.11 is similar to that of Theorem 6.9 we omit it.

Finally, we introduce the notion of transposition solutions of (6.13) which is equivalent to the notion of weak solution and which is required for the duality argument in Sect. 5. For that we introduce the space

$$\begin{aligned} \mathcal{R}_0 \stackrel{\text{def}}{=} \big\{&(y,\pi,\rho,\kappa,\mu) \in \mathbf{C}^2(\overline{Q}) \times C^1(\overline{Q}) \times \mathbf{C}^2(\overline{Q}) \times C^1(\overline{Q}) \times C([0,T])^N \mid \\ &y = 0,\ \rho \cdot n = 0 \ \text{ and } \ \operatorname{curl} \rho \times n = 0 \text{ on } \Sigma, \\ &\operatorname{div} y = \operatorname{div} \rho = 0 \ \text{ in } Q, \\ &\forall t \in [0,T] \quad \int_\Omega \rho(t) \cdot \mathrm{g}_i \mathrm{d}x = 0,\ i = 1, \ldots, N, \\ &\forall t \in [0,T] \quad \int_\Omega \pi(t) \mathrm{d}x = \int_\Omega \kappa(t) \mathrm{d}x = 0\big\}, \end{aligned} \tag{6.21}$$

and for $(y,\pi,\rho,\kappa,\mu)\in\mathcal{R}_0$ we set:

$$
\begin{aligned}
\mathcal{L}_1^*(y,\pi,\rho) &\stackrel{\text{def}}{=} -y_t-\Delta y-(D^s y)\overline{v}+(D^a\rho)\overline{B}+\nabla\pi,\\
\mathcal{L}_2^*(y,\kappa,\rho,\mu) &\stackrel{\text{def}}{=} -\rho_t-\Delta\rho+(D^s y)\overline{B}-(D^a\rho)\overline{v}+\nabla\kappa+\sum_{i=1}^N \mu_i \mathrm{g}_i.
\end{aligned}
\tag{6.22}
$$

Definition 6.12. Assume that $(w_0,\theta_0)\in\mathbf{V}_n^0(\Omega)\times\widehat{\mathbf{V}}_n^0(\Omega)$ and that $(g_1,g_2)\in L^2(0,T;\mathbf{H}^{-1}(\Omega))\times[\mathbf{H}_n^1(\Omega))]')$. We say that $(w,\theta)\in\mathbf{L}^2(Q)\times\mathbf{L}^2(Q)$ is a transposition solution of (1.19) if for all $(y,\pi,\rho,\kappa,\mu)\in\mathcal{R}_0$ such that $(y(T),\rho(T))=(0,0)$ the following equality holds:

$$
\begin{aligned}
&\int_Q (w\cdot\mathcal{L}_1^*(y,\pi,\rho)+\theta\cdot\mathcal{L}_2^*(y,\kappa,\rho,\mu))\mathrm{d}x\mathrm{d}\tau\\
&\quad=\int_0^T\langle(g_1,g_2)|(y,\rho)\rangle_{\mathcal{V}',\mathcal{V}}\mathrm{d}\tau+\int_\Omega(w_0\cdot y(0)+\theta_0\cdot\rho(0))\mathrm{d}x.
\end{aligned}
\tag{6.23}
$$

Proposition 6.13. *For $(w_0,\theta_0)\in\mathbf{V}_n^0(\Omega)\times\widehat{\mathbf{V}}_n^0(\Omega)$ and $(g_1,g_2)\in L^2(0,T;\mathbf{H}^{-1}(\Omega))\times[\mathbf{H}_n^1(\Omega))]')$, (w,θ) is a weak solution of (1.19) if and only if it is a transposition solution of (1.19).*

Proof. We first remark that the existence and the uniqueness of a transposition solution (w,θ) is a direct consequence of Theorem 6.11. Indeed, since a density argument guarantees that the transposition solution (w,θ) must satisfy for all $(f_1,f_2)\in\mathbf{L}^2(Q)\times\mathbf{L}^2(Q)$

$$
\int_Q(w\cdot f_1+\theta\cdot f_2)\mathrm{d}x\mathrm{d}\tau=\int_0^T\langle(g_1,g_2)|(y,\rho)\rangle_{\mathcal{V}',\mathcal{V}}\mathrm{d}\tau+\int_\Omega(w_0\cdot y(0)+\theta_0\cdot\rho(0))\mathrm{d}x,
$$

where $(y,\rho)\in W(0,T;\mathbf{V}_0^2(\Omega)\times\widehat{\mathbf{V}}_n^2(\Omega),\mathbf{V}_n^0(\Omega)\times\widehat{\mathbf{V}}_n^0(\Omega))$ is the corresponding solution of (6.19) with $(y_T,\rho_T)=(0,0)$, the conclusion follows from the Riesz representation theorem. Then it follows that weak solution and transposition solution coincide. Indeed, to get that every weak solution is a transposition solution it suffices to choose a smooth (y,ρ) in formulation (6.14) and to integrate by parts to recover formulation (6.23). Thus, the fact that every transposition solution is a weak solution follows from the uniqueness of the transposition solution. □

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