MAMERN09: 3^{rd} International Conference on Approximation Methods
and Numerical Modelling in Environment and Natural Resources
Pau (France), June 8-11, 2009

Riccati-based Strategies For Feedback Boundary Stabilization of the 3D Navier-Stokes Equations

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Keywords: Navier-Stokes equations, Feedback stabilization, Riccati equation.

Abstract. *The present paper is a synthesis of the two recent papers [1, 2] and of a forthcoming work. We deal with the local exponential stabilization of the Navier-Stokes equations in a bounded regular domain of* $\mathbb{R}^3$*, around a given steady-state flow, by means of a feedback boundary control. We consider the three differents situations where the control is described by a Dirichlet boundary condition, by a Neumann boundary condition, and by an evolution equation stated on the boundary of the domain (such a control is called dynamic control). We use a Riccati-based strategy to construct a linear feedback law for each type of boundary condition. Then we obtain a stable strong solution to the closed-loop Navier-Stokes equations and a corresponding Lyapunov function. We conclude that in the case of Dirichlet control the initial datum must fit a very specific trace condition, whereas no such restriction is required when considering Neumann or dynamic control.*

1 Introduction

Let us consider a regular trajectory (z, r) of the three dimensional Navier-Stokes equations:

$$\partial_t z - \nu \Delta z + (z \cdot \nabla) z + \nabla r = f, \quad \nabla \cdot z = 0 \quad \text{in} \quad (0, \infty) \times \Omega. \tag{1}$$

Here, Ω is a bounded and connected open subset of $\mathbb{R}^3$ with a boundary $\partial\Omega$ of class C^4, $\nu > 0$ is the viscosity coefficient and $f \in [H^1(\Omega)]^3$. For a given stationary trajectory (z_e, r_e) of (1) we look for a boundary condition in a feedback form $\mathcal{BC}(z, r, z_e, r_e) = 0$ on $\partial\Omega$ such that z obeys:

$$\lim_{t \to +\infty} z(t) = z_e, \tag{2}$$

at least if the initial velocity $z(0)$ is close to z_e. To achieve this goal, we first consider a nonhomogeneous boundary condition of type:

$$\mathcal{T}(z, r, z_e, r_e) = u \quad \text{on} \quad \partial\Omega, \tag{3}$$

for a control u in $V^0(\partial\Omega) := \{\xi \in [L^2(\partial\Omega)]^3 \mid \int_{\partial\Omega} \xi \cdot n = 0\}$, and then we use a Riccati-based strategy to obtain a boundary datum of the form $u = F(z - z_e)$ for some linear operator F defined on the space of divergence free vector fields $V^0(\Omega) := \{\xi \in [L^2(\Omega)]^3 \mid \nabla \cdot \xi = 0\}$ and with values in $V^0(\partial\Omega)$. Among all possible types of nonhomogeneous boundary conditions we choose to investigate the case of Dirichlet, of Neumann and of dynamic Dirichlet condition:

$$\text{(Dirichlet)} \quad \mathcal{T}(z, r, z_e, r_e) \quad := \quad z - z_e \quad \text{on} \quad \partial\Omega \tag{4}$$

$$\text{(Neumann)} \quad \mathcal{T}(z, r, z_e, r_e) \quad := \quad \frac{d(z - z_e)}{dn} - (r - r_e)n \quad \text{on} \quad \partial\Omega \tag{5}$$

$$\text{(Dynamic)} \quad \mathcal{T}(z, r, z_e, r_e) \quad := \quad \partial_t z - \Delta_b(z - z_e) - \sigma\, n \ \text{ on } \ \partial\Omega \tag{6}$$

$$\text{where} \quad \sigma = \int_{\partial\Omega} (\partial_t z - \Delta_b(z - z_e)) \cdot n. \tag{7}$$

In the above setting Δ_b denotes the vector-valued Laplace Beltrami operator, see [1, par. 5.].

The Riccati approach consists in reducing the stabilization problem (1)-(2)-(3) to the question of stabilizing around zero a system of type:

$$y' + Ay + N(y) = Bu, \tag{8}$$

where $y := Y(z - z_e)$ is a new state variable obtained from a linear mapping $Y : V^0(\Omega) \to H$ which sends $V^0(\Omega)$ into a well-chosen Hilbert space H, where A and B are adequate linear operators, and where $N(\cdot)$ is a nonlinear mapping obeying $N'(0) = 0$. Thus, we linearize (8) around zero, we state a minimization problem on the resulting linear system, and we obtain an optimal control in the feedback form $u = -B^*\Pi y$ where the linear mapping Π is the unique solution to an Algebraic Riccati equation. Finally, we verify that such a feedback control stabilizes (8) around (z_e, r_e), which means that the pair (z, r) solution to (1) and

$$\mathcal{T}(z, r, z_e, r_e) = -B^*\Pi Y(z - z_e) \quad \text{on} \quad \partial\Omega, \tag{9}$$

obeys (2) when $z(0) - z_e$ is small. The next section details the main steps of the method.

2 General framework of the Riccati approach

Let H be a Hilbert space, let A be a closed linear operator in H, with domain $\mathcal{D}(A)$, and which is the infinitesimal generator of an analytic semigroup $(e^{-At})_{t\geq 0}$ on H. We define $\widehat{A} = A + \lambda$ for $\lambda > 0$ large enough so that the fractional powers of $\widehat{A}$ are well-defined, and we assume that the complex interpolation equality $\mathcal{D}(\widehat{A}^\theta) = [\mathcal{D}(A), H]_{1-\theta}$ holds for all $\theta \in [0, 1]$. Moreover, A^* with domain $\mathcal{D}(A^*)$ denotes the adjoint of A and $\mathcal{D}(A^*)'$ denotes the dual space of $\mathcal{D}(A^*)$ with respect to the pivot space H. We also introduce a linear operator $B : U \to \mathcal{D}(A^*)'$, defined on a Hilbert space U, and such that $\widehat{A}^{-\gamma}B : U \to H$ is bounded for $0 \leq \gamma < 1$. Finally, for two Hilbert spaces X, Y, we denote by $L^2(X) := L^2(0, \infty; X)$ and $H^1(Y) := H^1(0, \infty; Y)$ the usual vector-valued Lebesgue and Sobolev spaces, and we set $W(X, Y) = L^2(X) \cap H^1(Y)$.

For an initial datum $y_0 \in H$ and a control function $u \in L^2(U)$ we consider the solution $y \in W(H, \mathcal{D}(A^*)')$ to the following linearization around zero of system (8):

$$y' + Ay = Bu \ \in \mathcal{D}(A^*)', \quad y(0) = y_0. \tag{10}$$

The Riccati approach consists in checking the well-posedness and then solving:

$$\inf \Big\{ \int_0^\infty \|y(t)\|_H^2 dt + \int_0^\infty \|u(t)\|_U^2 dt \ \mid \ (y, u) \in W(H, \mathcal{D}(A^*)') \times L^2(U) \ \text{ obeys } \ (10) \Big\},$$

whose optimal pair $(\widehat{y},\widehat{u})$ obeys the feedback equality $\widehat{u}(t) = -B^*\Pi\widehat{y}(t)$, where the linear mapping Π is bounded from H into $\mathcal{D}(A^*)$ and is the unique solution to the Riccati equation:

$$\Pi^* = \Pi > 0 \quad \text{and} \quad A^*\Pi + \Pi A + \Pi B B^*\Pi = I.$$

Then $\widehat{y}$ obeys $\widehat{y}' + A_\Pi\widehat{y} = 0$, as well as the semigroup formula $\widehat{y}(t) = e^{-A_\Pi t}y_0$, where

$$\mathcal{D}(A_\Pi) = \{\xi \in H \mid A\xi + B(B^*\Pi)\xi \in H\} \quad \text{and} \quad A_\Pi\xi = A\xi + B(B^*\Pi)\xi.$$

The next proposition states that $\widehat{y}(t) \to 0$ as $t \to \infty$ and gives some useful properties of A_Π.

Proposition 1. *The closed-loop operator $(\mathcal{D}(A_\Pi), -A_\Pi)$ is the infinitesimal generator of an analytic and exponentially stable semigroup on H, the adjoint of $(\mathcal{D}(A_\Pi), A_\Pi))$ is given by*

$$\mathcal{D}(A_\Pi^*) = \mathcal{D}(A^*) \quad \textit{and} \quad A_\Pi^* = A^* + (B^*\Pi)^*B^*,$$

and for all $\theta \in [0,1]$ we have $\mathcal{D}(A_\Pi^{*\theta}) = \mathcal{D}(\widehat{A}^{*\theta})$ and $\mathcal{D}(A_\Pi^\theta) = [\mathcal{D}(A_\Pi), H]_{1-\theta}$.

The above proposition allows to define $\Pi^{(s)} := A_\Pi^{*s/2+1/2}\,\Pi\,A_\Pi^{s/2+1/2}$ for $s \in [0,1]$ as a bounded operator from $\mathcal{D}(A_\Pi^{s/2})$ into $\mathcal{D}(A_\Pi^{s/2})'$, and to introduce the following norms:

$$\|\xi\|_{\Pi,s} = \sqrt{(\xi|\xi)_{\Pi,s}} \quad \text{where} \quad (\xi|\zeta)_{\Pi,s} := \langle \Pi^{(s)}\xi|\zeta\rangle_{\mathcal{D}(A_\Pi^{s/2})',\mathcal{D}(A_\Pi^{s/2})},$$

$$\|\xi\|_{\Pi,s+1} = \sqrt{(A_\Pi\xi|\xi)_{\Pi,s}} \quad \text{and} \quad \|\xi\|_{\Pi,s-1} = \sup_{\zeta\in\mathcal{D}(A_\Pi^{s/2+1/2})} \frac{(\xi|\zeta)_{\Pi,s}}{\|\zeta\|_{\Pi,s+1}},$$

which define equivalent topologies for $\mathcal{D}(A_\Pi^{s/2})$, $\mathcal{D}(A_\Pi^{s/2+1/2})$ and $\mathcal{D}(\widehat{A}^{*1/2-s/2})'$ respectively. Thus, for $y_0 \in \mathcal{D}(A_\Pi^{s/2})$ we prove that $t \longmapsto \|e^{-A_\Pi t}y_0\|_{\Pi,s}$ decreases with an exponential rate:

$$\beta_s = \inf_{0\neq\xi\in\mathcal{D}(A_\Pi^{1/2+s/2})} \frac{\|\xi\|^2_{\Pi,s+1}}{\|\xi\|^2_{\Pi,s}},$$

and that the nonhomogeneous system $y' + A_\Pi y = f$ and $y(0) = y_0$ admits a unique solution in $W_s := W(\mathcal{D}(A_\Pi^{1/2+s/2}), \mathcal{D}(\widehat{A}^{*1/2-s/2})')$ when $f \in L^2(\mathcal{D}(\widehat{A}^{*1/2-s/2})')$. Hence, if we consider:

$$y' + A_\Pi y + N(y) = 0, \quad y(0) = y_0, \tag{11}$$

for a nonlinear mapping $N(\cdot)$ obeying:

$$(\mathcal{L}_s)\left\{\begin{array}{rcl} \|N(\xi)\|_{\Pi,s-1} & \leq & C\|\xi\|_{\Pi,s}\|\xi\|_{\Pi,s+1}, \\ \|N(\xi)-N(\zeta)\|_{\Pi,s-1} & \leq & C(\|\xi-\zeta\|_{\Pi,s}\|\xi\|_{\Pi,s+1} + \|\zeta\|_{\Pi,s}\|\xi-\zeta\|_{\Pi,s+1}), \end{array}\right.$$

for some $s \in [0,1]$, then $(\mathcal{L}_s)$ combined with the above existence result for the nonhomogeneous closed-loop linear system provides a fixed point solution to (11).

Theorem 1. *Assume $(\mathcal{L}_s)$ for $s \in [0,1]$ and $y_0 \in \mathcal{D}(A_\Pi^{s/2})$.*
There is $\rho > 0$ and $\mu > 0$ such that, if $\delta \in (0,\mu)$ and $\|y_0\|_{\Pi,s} < \rho\delta$, then system (11) admits a solution $y_{y_0} \in W_s$ such that $\|y_{y_0}\|_{W_s} \leq \delta$, which is unique within the class of functions in $L^\infty_{loc}(\mathcal{D}(A_\Pi^{s/2})) \cap L^2_{loc}(\mathcal{D}(A_\Pi^{1/2+s/2}))$. Moreover, every solution with an initial datum obeying:

$$\|y_0\|_{\Pi,s} < R_s \quad \textit{where} \quad \frac{1}{R_s} = \sup_{0\neq\xi\in\mathcal{D}(A_\Pi^{1/2+s/2})} \frac{(N(\xi)|\xi)_{\Pi,s}}{\|\xi\|_{\Pi,s}\|\xi\|^2_{\Pi,s+1}},$$

is such that $t \longmapsto \|y_{y_0}(t)\|_{\Pi,s}$ is decreasing and $\|y_{y_0}(t)\|_{\Pi,s} \leq \|y_0\|_{\Pi,s}e^{-\beta_s(1-\|y_0\|_{\Pi,s}/R_s)t}$.

Finally, to obtain a solution to (1)-(2)-(9) it suffices to interpret (11) and Theorem 1 in terms of partial differential equations.

3 Feedback Boundary Stabilization of the 3D Navier-Stokes Equations

In each of cases (4), (5), (6) there is a mapping $Y : V^0(\Omega) \to H$ and a system (10) for which the Riccati approach provides operators Π and A_Π such that Proposition 1 and Theorem 1 for $s \in (1/2, 1]$ hold. It yields Theorem 2 below, which states the existence of a stable solution $(z, r) \in \{(z_e, r_e)\} + V^{1+s,1/2+s/2} \times \mathbb{H}^{s,s/2-1/2}$ to (1)-(9) when $z(0) - z_e$ is small in some subspace $V^s_\Pi(\Omega) \subset Y^{-1}[\mathcal{D}(A_\Pi^{s/2})]$. Theorem 2 uses notations $V^{\alpha,\beta} := L^2(0, \infty; V^0(\Omega) \cap [H^\alpha(\Omega)]^3) \cap H^\beta(0, \infty; V^0(\Omega))$, $\mathbb{H}^{\alpha,\beta} := H^\beta(0, \infty; H^\alpha(\Omega))$ for $\alpha \geq 0$, $\beta \in \mathbb{R}$, and $V^{\alpha,\beta}_{loc}$, $\mathbb{H}^{\alpha,\beta}_{loc}$ which denote analogous locally time integrable spaces. The restriction $s \in (1/2, 1]$ is due to the behaviour of the 3D Navier-Stokes' nonlinearity. It implies that $\mathcal{D}(A_\Pi^{s/2})$ and $V^s_\Pi(\Omega)$ are significantly strict subspaces of H and $V^0(\Omega)$ respectively. For instance, in the case of Dirichlet control, where $Y := P$ is the orthogonal projector from $V^0(\Omega)$ onto $V^0_n(\Omega) := \{\xi \in V^0(\Omega) \mid \xi \cdot n = 0 \text{ on } \partial\Omega\}$ and where the corresponding input operator B has a high degree of unboundedness $\gamma = 3/4 + \epsilon$, the elements of $V^s_\Pi(\Omega) \subset P^{-1}[\mathcal{D}(A_\Pi^{s/2})]$ with $s \in (1/2, 1]$ obey a very specific trace condition:

$$V^s_\Pi(\Omega) = \Big\{\xi \in [H^s(\Omega)]^3 \mid \nabla \cdot \xi = 0,\ \xi = \nu \frac{d}{dn}\Big(\Pi P \xi\Big) - \psi n \text{ on } \partial\Omega,\ \psi \text{ obeys } (\mathcal{N}_\xi)\Big\},$$

$$\text{where} \quad (\mathcal{N}_\xi) \begin{cases} \Delta\psi = \nabla \cdot (z_e \cdot \nabla - (\nabla z_e)^T)\Pi P \xi \ \text{ in } \Omega, \ \displaystyle\int_{\partial\Omega} \psi = 0, \\ \dfrac{d\psi}{dn} = [(\nu\Delta - (\nabla z_e)^T + z_e \cdot \nabla)\Pi P \xi] \cdot n \ \text{ on } \partial\Omega. \end{cases}$$

Neumann and dynamic control are less restrictive because they correspond to a B with a low degree of unboundedness and then no specific trace condition is imposed on initial data: $Y\xi := \xi \in V^0(\Omega)$ and $\gamma = 1/4 + \epsilon$ for the Neumann case while $Y\xi = (P\xi, \xi|_{\partial\Omega}) \in V^0_n(\Omega) \times V^0(\partial\Omega)$ and $\gamma = 0$ for the dynamic case, and in both situations $V^s_\Pi(\Omega) = \{\xi \in [H^s(\Omega)]^3 \mid \nabla \cdot \xi = 0\}$ with $s \in (1/2, 1]$. Finally, for each type of control the expression of (9) is:

$$\text{(Dirichlet)} \qquad z - z_e = \nu \frac{d}{dn}\Big(\Pi P(z - z_e)\Big) - \psi\, n \ \text{ on } \partial\Omega, \quad \psi \text{ obeys } (\mathcal{N}_{z-z_e}), \tag{12}$$

$$\text{(Neumann)} \qquad \nu \frac{d(z - z_e)}{dn} - (r - r_e)n = \Pi(z_e - z) \text{ on } \partial\Omega, \tag{13}$$

$$\text{(Dynamic)} \qquad \partial_t z - \Delta_b(z - z_e) - \sigma\, n = \Pi_1 P(z_e - z) + \Pi_2(z_e - z) \ \text{ on } \partial\Omega, \tag{14}$$

where σ obeys (7) and $\Pi_1 : V^0(\Omega) \to V^0(\partial\Omega)$, $\Pi_2 : V^0(\partial\Omega) \to V^0(\partial\Omega)$ are components of Π.

Theorem 2. *For $s \in (1/2, 1]$, there is $\rho > 0$ and $\mu > 0$ such that, if $\delta \in (0, \mu)$ and $z(0) - z_e \in V^s_\Pi(\Omega)$ obeys $\|Y(z(0) - z_e)\|_{\Pi,s} \leq \rho\delta$ then (1), (12) [Resp. (13) and (14)] admits a solution (z, r) in $\{(z_e, r_e)\} + V^{1+s,1/2+s/2} \times \mathbb{H}^{s,s/2-1/2}$ such that $\|z - z_e\|_{V^{1+s,1/2+s/2}} \leq \delta$ and $\|r - r_e\|_{\mathbb{H}^{s,s/2-1/2}} \leq \delta(1 + \delta)$, which is unique within the class of functions in $\{(z_e, r_e)\} + V^{1+s,1/2+s/2}_{loc} \times \mathbb{H}^{s,s/2-1/2}_{loc}$. Moreover, every solution obeying $\|Y(z(0) - z_e)\|_{\Pi,s} < R_s$ is such that $t \mapsto \|Y(z(t) - z_e)\|_{\Pi,s}$ is decreasing and $\|Y(z(t) - z_e)\|_{\Pi,s} \leq \|Y(z(0) - z_e)\|_{\Pi,s} e^{-\beta_s(1 - \|Y(z(0) - z_e)\|_{\Pi,s}/R_s)t}$.*

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