GLOBAL CARLEMAN INEQUALITIES FOR STOKES AND PENALIZED STOKES EQUATIONS

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Abstract. In this note we use the result of [22] to prove a global Carleman inequality related to the null controllability of penalized Stokes kind systems. The constants of the obtained Carleman inequality are uniform in terms of the penalization parameter ε. It then provides a null control with a uniformly (in ε) bounded L^2 norm. With a limiting argument we also deduce a new Carleman inequality for Stokes type system. Thus, we apply theses results to obtain the null controllability of Oseen and Navier-Stokes system in the penalized and in the non penalized cases.

1. **Introduction.** In the present paper we are interested in proving the uniform (in ε) null controllability of the following penalized Stokes kind system:

$$\left\{\begin{array}{rcll} \partial_t v - \Delta v + B(z,v) + B(v,z) - \dfrac{1}{\varepsilon}\nabla(\nabla\cdot v) &=& h\mathbf{1}_{\mathcal{O}} & \text{in } \ \Omega\times(0,T),\\ v &=& 0 & \text{on } \ \partial\Omega\times(0,T),\\ v(0) &=& v_0 & \text{in } \ \Omega. \end{array}\right. \tag{1}$$

Here $\Omega\subset\mathbb{R}^d$ is a bounded domain of class C^2, $d=2$ or $d=3$, $\mathcal{O}\subset\Omega$ is a nonempty open subset, $\mathbf{1}_{\mathcal{O}}$ is the characteristic function of $\mathcal{O}$, T is a given positive time, $\varepsilon>0$ is a small parameter, $z\in L^\infty(0,T;(W^{1,\infty}(\Omega))^d)$ and $B(\cdot,\cdot)$ denotes the following bilinear mapping:

$$B(w_1,w_2) = (w_1\cdot\nabla)w_2 + \frac{1}{2}(\nabla\cdot w_1)w_2.$$

In (1), $h\in(L^2(\Omega\times(0,T)))^d$ is the control function. We recall that system (1) is said to be null controllable in time T, if and only if, for all $v_0\in(L^2(\Omega))^d$ there exists $h\in(L^2(\Omega\times(0,T)))^d$ such that the solution to (1) obeys:

$$v(T)=0 \quad \text{in } \ \Omega. \tag{2}$$

Moreover, we will say that (1) is uniformly (in ε) null controllable in time T, if for all $v_0\in(L^2(\Omega))^d$ there exists a null contol h (steering the velocity from $v(0)=v_0$ to (2)) with norm in $(L^2(\Omega\times(0,T)))^d$ bounded independently on $\varepsilon>0$.

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System (1) is obtained by approximating the linearized (about z) Navier-Stokes system with the penalty method. It consists in replacing the vanishing divergence condition

$$\nabla \cdot y = 0 \quad \text{in} \quad \Omega \times (0,T),$$

by the penalized condition:

$$\nabla \cdot y + \varepsilon p = 0 \quad \text{in} \quad \Omega \times (0,T). \tag{3}$$

Such a method has been first introduced in [25] for the Navier-Stokes system. The denomination *penalized* comes from the fact that, when dealing with the approximation of the stationary Stokes equations, (3) is obtained from a minimization problem where the free divergence condition is no more imposed as a constraint but only penalized, see [13, Rem. 4.4, p.67]. The question of the uniform null controllability of (1) is interesting for applications, in particular to study the approximation by the penalty method of some control problems. For instance, it may allow to define optimal control problems stated over an infinite time horizon which provide uniformly (in ε) stabilizing feedback law for penalized Oseen and Navier-Stokes system (about non penalized problem see [24, 3, 2]). Moreover, the uniform null controllability of (1) may also allow to extend to infinite time horizon problem the error analysis made in [1] in the case of a finite time horizon problem.

Let us recall that one of the most popular strategy to obtain the null controllability of parabolic systems is the use of global Carleman inequalities which have been introduced in [5, 6, 12]. Such a type of inequality is a sophisticated weighted estimate which allows to control the values of the solution in the whole domain Ω, in terms of its values in a subdomain $\mathcal{O} \subset\subset \Omega$. Global Carleman inequalities have been intensively used to obtain the null controllability of Stokes kind system. For Stokes and Navier-Stokes equations see [11, 8, 20, 10, 4, 18], and about some other related systems see [19, 16, 14, 17, 7]. The goal of the present paper is to provide a Carleman inequality related to the uniform null controllability of (1). Thus, with a limiting argument when $\varepsilon \to 0^+$, we obtain a new global Carleman inequality for Stokes type system which improves the ones of [8] and of [17, App.], and which can be used to simplify the proof of some known null controllability results such as the ones of [7, 16].

It is well-known that the null controllability of (1) is equivalent to an observability inequality satisfied by an adjoint system: system (1) is null controllable in time T, if and only if, there exists $C_T > 0$ such that for all $y_T \in (L^2(\Omega))^d$ the solution to:

$$\left\{ \begin{array}{rcll} -\partial_t y - \Delta y - (\nabla y)z - {}^t(\nabla y)z - \frac{1}{2}(\nabla \cdot z)y & & & \\ +\frac{1}{2}\nabla(z\cdot y) - \frac{1}{\varepsilon}\nabla(\nabla\cdot y) & = & 0 & \text{in } \Omega\times(0,T), \\ y & = & 0 & \text{on } \partial\Omega\times(0,T), \\ y(T) & = & y_T & \text{in } \Omega, \end{array} \right. \tag{4}$$

satisfies the observability inequality

$$\|y(0)\|^2_{(L^2(\Omega))^d} \le C_T \int_0^T \int_{\mathcal{O}} |y|^2 dx dt. \tag{5}$$

Moreover, for a given initial datum v_0 the norm of the smallest null control h in $(L^2(\Omega\times(0,T)))^d$ is bounded by $\sqrt{C_T}\|v_0\|_{(L^2(\Omega))^d}$. As a consequence, if the above observability inequality stands for a constant C_T which is independant on ε, then (1) is uniformly (in ε) null controllable in time T. A way to obtain such a uniform inequality is to prove a global Carleman inequality related to system (4) with constants independent on ε. For details about the way one deduces the observability inequality (5) from a global Carleman inequality, or about the equivalence between null controllability and observability of the adjoint system, see the introduction of [9]. Here we only focus on the proof of a uniform global Carleman inequality which is provided in Theorem 2.2 below. We shall underline that our result heavily relies on the use of a global Carleman inequality for parabolic equations with nonhomogeneous Dirichlet boundary conditions which has been recently proved in [22]. Note also that if $\nabla\cdot z=0$ in $\Omega\times(0,T)$ then the Carleman inequality for the Stokes equation stated in Theorem 2.4 below can be used to obtain the null controllability of the linearized Navier-Stokes equation:

$$\left\{\begin{array}{rcll} \partial_t v-\Delta v+(z\cdot\nabla)v+(v\cdot\nabla)z+\nabla p &=& h\mathbf{1}_{\mathcal{O}} & \text{in } \Omega\times(0,T),\\ \nabla\cdot v &=& 0 & \text{in } \Omega\times(0,T),\\ v &=& 0 & \text{on } \partial\Omega\times(0,T),\\ v(0) &=& v_0 & \text{in } \Omega, \end{array}\right. \tag{6}$$

for $z\in L^\infty(0,T;(W^{1,\infty}(\Omega))^d)$, or for $z\in(L^\infty(\Omega\times(0,T)))^d$ and $\|z\|_{(L^\infty(\Omega\times(0,T)))^d}$ small enough. We also explain which improvement of [22, Thm. 2.1] would permit to relax the assumption $z\in L^\infty(0,T;(W^{1,\infty}(\Omega))^d)$ to $z\in(L^\infty(\Omega\times(0,T)))^d$ without assuming $\|z\|_{(L^\infty(\Omega\times(0,T)))^d}$ to be small. That would improve the known null controllability results stated in [8, 9, 20].

The rest of the paper is organized as follows. Section 2 is dedicated to notations and general definitions, and of the statement of our main results. Section 3 is devoted to applications. We apply our Carleman inequalities to penalized Oseen and Oseen equations in Subsection 3.1 to obtain the null controllability of (1) and (6), and we also discuss the required regularity for z. In Subsection 3.2 we recover in a very simple way a null controllability result obtained in [7] for a linear micropolar fluid system. In Subsection 3.3 we state local null controllability results for penalized Navier-Stokes and Navier-Stokes systems. Finally, we prove preliminary lemmas in Section 4 and the proofs of our theorems are postponed to Sections 5, 6 and 7.

2. **Definitions and main results.** Let us recall that $\Omega\subset\mathbb{R}^d$, $d=2,3$, is a bounded open subset of class C^2. We denote by $n={}^t(n_1,\dots,n_d)$ the unit interior normal vector field defined on $\partial\Omega$ (^{t} denotes the transpose). For a scalar function w or a vector field $y={}^t(y_1,\dots,y_d)$ we define $\nabla w={}^t(\partial_{x_1}w,\dots,\partial_{x_d}w)$, $\nabla y=(\partial_{x_j}y_i)_{1\le i,j\le d}$, ${}^t\nabla y=(\partial_{x_i}y_j)_{1\le i,j\le d}$ and the normal derivatives $\frac{dw}{dn}=(\nabla w)\cdot n$ and $\frac{dy}{dn}=(\nabla y)n$ on $\partial\Omega$. We recall that the divergence of y is defined by $\nabla\cdot y=\sum_{j=1}^d\partial_{x_j}y_j$ and the curl of y or w is defined by

$$\nabla\times y=\partial_{x_1}y_2-\partial_{x_2}y_1 \quad \text{and} \quad \nabla\times w=\begin{pmatrix}\partial_{x_2}w\\ -\partial_{x_1}w\end{pmatrix} \quad \text{if } d=2,$$

and

$$\nabla \times y = \begin{pmatrix} \partial_{x_2} y_3 - \partial_{x_3} y_2 \\ \partial_{x_3} y_1 - \partial_{x_1} y_3 \\ \partial_{x_1} y_2 - \partial_{x_2} y_1 \end{pmatrix} \quad \text{if } d = 3.$$

Moreover, we set $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$ and for $r \geq 0$ we use the notations:

$$\begin{array}{rcl} H^{\frac{r}{2},r}(Q) & = & H^{\frac{r}{2}}(0,T;L^2(\Omega)) \cap L^2(0,T;H^r(\Omega)), \\ H^{\frac{r}{2},r}(\Sigma) & = & H^{\frac{r}{2}}(0,T;L^2(\partial\Omega)) \cap L^2(0,T;H^r(\partial\Omega)). \end{array}$$

In order to state our main result we need first to introduce some usual weight functions. Let $\omega \subset \mathcal{O}$ a nonempty open subset such that $\overline{\omega} \subset \mathcal{O}$ and let $\eta \in C^2(\overline{\Omega})$ such that

$$\begin{array}{rcll} \eta(x) & = & 0 & \forall x \in \partial\Omega, \\ \eta(x) & > & 0 & \forall x \in \Omega, \\ |\nabla \eta(x)| & > & 0 & \forall x \in \overline{\Omega \backslash \omega}. \end{array}$$

For the existence of such a function see for instance [12]. Thus, we introduce $e \in C^\infty([0,1])$ such that $e(t) = t$ for $t \in (0,1/4)$, $e(t) = 1-t$ for $t \in (3/4,1)$ and $e(t) \in [1/4,1/2]$ for $t \in (1/4,3/4)$, and we define $\ell(t) = Te(t/T)$. It is obvious to see that:

$$\ell \in C^\infty([0,T]) \quad \text{and} \quad \begin{cases} \ell(t) = t & \forall t \in (0,T/4), \\ \ell(t) \in [T/4,T/2] & \forall t \in (T/4,3T/4), \\ \ell(t) = T-t & \forall t \in (3T/4,T). \end{cases} \tag{7}$$

Finally, for $k \geq 2$ and $\lambda > 1$ we define:

$$\alpha(x,t) = \frac{e^{\lambda\eta(x)} - e^{2\lambda\|\eta\|_\infty}}{\ell(t)^k}, \qquad \varphi(x,t) = \frac{e^{\lambda(\eta(x)+\|\eta\|_\infty)}}{\ell(t)^k}, \tag{8}$$

and

$$\widehat{\alpha}(t) = \min_{x\in\overline{\Omega}} \alpha(x,t) = \frac{1 - e^{2\lambda\|\eta\|_\infty}}{\ell(t)^k}, \qquad \widehat{\varphi}(t) = \min_{x\in\overline{\Omega}} \varphi(x,t) = \frac{e^{\lambda\|\eta\|_\infty}}{\ell(t)^k}, \tag{9}$$

and

$$\alpha^*(t) = \max_{x\in\overline{\Omega}} \alpha(x,t) = \frac{e^{\lambda\|\eta\|_\infty} - e^{2\lambda\|\eta\|_\infty}}{\ell(t)^k}, \qquad \varphi^*(t) = \max_{x\in\overline{\Omega}} \varphi(x,t) = \frac{e^{2\lambda\|\eta\|_\infty}}{\ell(t)^k}.$$

The key tool that is used in the present paper is the following Lemma which is proved in [22].

Lemma 2.1. *Let $F_0 \in L^2(Q)$ and $F_1 \in (L^2(Q))^d$ and let $\psi \in H^{\frac{1}{2},1}(Q)$ satisfying*

$$-\partial_t \psi - \Delta\psi = \nabla \cdot F_1 + F_0 \quad \text{in } Q.$$

There exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ there exists two constants $c_0(\lambda) > 1$ and $s_0(\lambda) > 0$, such that for all $s \geq s_0(\lambda)$, the following inequality holds:

$$\begin{aligned} & s^{-1} \int_Q e^{2s\alpha} \varphi^{-1} |\nabla\psi|^2 dx d\tau + s\lambda^2 \int_Q e^{2s\alpha} \varphi |\psi|^2 dx d\tau \leq \\ & \quad c_0(\lambda) \Big(s^{-2}\lambda^{-2} \int_Q e^{2s\alpha} \varphi^{-2} |F_0|^2 dx d\tau + \int_Q e^{2s\alpha} |F_1|^2 dx d\tau \\ & \qquad + s\lambda^2 \int_0^T \int_\omega e^{2s\alpha} \varphi |\psi|^2 dx d\tau + s^{-\frac{1}{2}} \|\widehat{\varphi}^{-\frac{1}{4}} \psi e^{s\widehat{\alpha}}\|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} \Big). \end{aligned}$$

Remark 1. *(i)*The Carleman inequality provided in [22, Thm.2.1] involves the additional boundary term $\|\widehat{\varphi}^{-\frac{1}{4}+\frac{1}{k}}\psi e^{s\widehat{\alpha}}\|^2_{L^2(\Sigma)}$ at the right side of the inequality. However, such a term can be dropped. Indeed, a careful checking of the proof of [22, Thm. 2.1] shows that the additional term $\|\widehat{\varphi}^{-\frac{1}{4}+\frac{1}{k}}\psi e^{s\widehat{\alpha}}\|^2_{L^2(\Sigma)}$ can be improved to $\|\widehat{\varphi}^{-\frac{1}{4}+\frac{1}{4k}}\psi e^{s\widehat{\alpha}}\|^2_{L^2(\Sigma)}$ and then be absorbed by $\|\widehat{\varphi}^{-\frac{1}{4}}\psi e^{s\widehat{\alpha}}\|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)}$ by using a Hardy-type inequality.
(ii) Since $c_0(\lambda)$ is not known, the explicit presence of the parameter λ in the Carleman inequality of Lemma 2.1 may seem useless. Note also that the inequality provided in [22, Thm.2.1] is not explicit in λ. However, writing the explicit dependence in λ permits to underline the role of $c_0(\lambda)$ in the Carleman inequalities obtained from Lemma 2.1, see Theorems 2.2 2.3, 2.4, 2.5 below. In particular, in section 3.1 below we show that if $c_0(\lambda)\lambda^{-2} \to 0$ as $\lambda \to +\infty$ then the Carleman inequality of Theorem 2.4 can be used to obtain the null controllability of (6) with $z \in (L^\infty(Q))^d$ only, instead of $z \in L^\infty(0,T;(W^{1,\infty}(\Omega))^d)$ or $z \in (L^\infty(Q))^d$ and $\|z\|_\infty$ small. In fact, in view of analogous inequality for a homogeneous boundary condition [23, 9], and for elliptic equation with a nonhomogeneous boundary condition [21], we conjecture that $c_0(\lambda)$ is bounded independently on λ. However, up to our knowledge, determining the asymptotic behaviour of $\lambda \mapsto c_0(\lambda)$ is still an open question.

Next, for
$$y_T \in (L^2(\Omega))^d, \tag{10}$$
we consider the nonhomogeneous system:
$$\begin{cases} -\partial_t y - \Delta y - \dfrac{1}{\varepsilon}\nabla(\nabla\cdot y) = f & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(T) = y_T & \text{in } \Omega, \end{cases} \tag{11}$$
and we first suppose that f has the following general form:
$$f = f_0 + f_1 + \sum_{j=1}^{n} \left((\nabla f_{1,j}) z_{1,j} + {}^t(\nabla f_{2,j}) z_{2,j} \right), \quad n \in \mathbb{N}, \tag{12}$$
where for $i = 1, 2$ and $j = 1, \ldots, n$:
$$\begin{gathered} f_i \in (L^2(Q))^d, \quad \nabla \times f_1 \in (L^2(Q))^{2d-3}, \\ f_{i,j} \in L^2(0,T;(H^1(\Omega))^d), \quad z_{i,j} \in L^\infty(0,T;(W^{1,\infty}(\Omega))^d). \end{gathered} \tag{13}$$
Moreover, let us introduce the following integral quantities:
$$\begin{aligned} I_0(s,\lambda,y) =& s^{-1}\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1}|\partial_t y|^2 dx d\tau + s^{-1}\int_Q e^{2s\alpha}\varphi^{-1}|\Delta y|^2 dx d\tau \\ &+ \int_Q e^{2s\alpha}|\nabla(\nabla\times y)|^2 dx d\tau + s\lambda^2\int_Q e^{2s\alpha}\varphi|\nabla y|^2 dx d\tau \\ &+ s^2\lambda^2\int_Q e^{2s\alpha}\varphi^2|\nabla\times y|^2 dx d\tau + s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|y|^2 dx d\tau, \end{aligned} \tag{14}$$

and

$$\begin{aligned} I_1(s,\lambda,f) =& s\int_Q e^{2s\alpha}\varphi|f_0|^2 dxd\tau \\ &+ s^{\frac{1}{2}}\int_Q e^{2s\alpha}\varphi|f_1|^2 dxd\tau + s^{-1}\lambda^{-2}\int_Q e^{2s\alpha}\varphi^{-1}|\nabla\times f_1|^2 dxd\tau \\ &+ \sum_{j=1}^n\sum_{i=1}^2\int_Q e^{2s\alpha}(s^{\frac{1}{2}}\varphi|\nabla f_{i,j}|^2 + s\varphi|\nabla\times f_{i,j}|^2)dxd\tau. \end{aligned} \tag{15}$$

In the following we use the notation $g(\lambda)\asymp h(\lambda)$ to say that g, h are asymptotically comparable, i.e there exist $0<c_1<c_2$, independent on λ, such that $c_1 g(\lambda)\le h(\lambda)\le c_2 g(\lambda)$ for all $\lambda>1$. The main result of the present paper is the following

Theorem 2.2. *Let $\varepsilon_0>0$, and assume (10), (12), (13) and $k\ge 4$. There exists $\lambda_0>0$ such that for all $\lambda\ge\lambda_0$ there exist two constants $\widehat{c}_0(\lambda)>1$ and $\widehat{s}_0(\lambda)>0$ such that for all $s\ge\widehat{s}_0(\lambda)$ and all $\varepsilon\in(0,\varepsilon_0)$ the solution to (11) obeys:*

$$\begin{aligned} I_0(s,\lambda,y)+&s^{-1}\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1}\frac{1}{\varepsilon^2}|\nabla(\nabla\cdot y)|^2 dxd\tau + s\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}|\varepsilon^{-\frac{1}{2}}\nabla\cdot y|^2 dxd\tau \\ &\le \widehat{c}_0(\lambda)I_1(s,\lambda,f) + \widehat{c}_0(\lambda)^2 s^4\lambda^4\int_0^T\int_{\mathcal{O}} e^{2s\alpha}\varphi^3|y|^2 dxd\tau. \end{aligned} \tag{16}$$

We have $\widehat{c}_0(\lambda)\asymp c_0(\lambda)$ and $\widehat{s}_0(\lambda)\asymp \max(s_0(\lambda), c_0(\lambda)^2\lambda^{-8}e^{2\lambda(1-\frac{1}{k})\|\eta\|_\infty})$ for $c_0(\lambda)$, $s_0(\lambda)$ given in Lemma 2.1. Note that the implicit constants of $\asymp$ depend on $\|z_{i,j}\|_{L^\infty(0,T;(W^{1,\infty}(\Omega))^d}$, $i=1,2$, $j=1,\dots,n$.

Suppose now that f in (11) is of the form:

$$f = F_0 + \nabla\cdot F_1 \quad \text{with} \quad F_0\in(L^2(Q))^d, \quad F_1\in(L^2(Q))^{d\times d}. \tag{17}$$

Then from Theorem 2.2 we deduce the following

Theorem 2.3. *Let $\varepsilon_0>0$, and assume (10), (17) and $k\ge 4$. There exists $\lambda_0>0$ such that for all $\lambda\ge\lambda_0$ there exist two constants $\widehat{c}_0(\lambda)>1$ and $\widehat{s}_0(\lambda)>0$ such that for all $s\ge\widehat{s}_0(\lambda)$ and all $\varepsilon\in(0,\varepsilon_0)$ the solution to (11) obeys:*

$$\begin{aligned} &\varepsilon s\lambda^2\int_Q e^{2s\alpha}\varphi|\nabla y|^2 dxd\tau + s\lambda^2\int_Q e^{2s\alpha}\varphi|\nabla\cdot y|^2 dxd\tau \\ &+ s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|y|^2 dxd\tau \le \widehat{c}_0(\lambda)^2\bigg(s\int_Q e^{2s\alpha}\varphi|F_0|^2 dxd\tau \\ &+ \frac{s^3\lambda^2}{\varepsilon}\int_Q e^{2s\alpha}\varphi^3|F_1|^2 dxd\tau + s^4\lambda^4\int_0^T\int_{\mathcal{O}} e^{2s\alpha}\varphi^3|y|^3 dxd\tau\bigg). \end{aligned} \tag{18}$$

and

$$\begin{aligned} &s^2\lambda^2\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}|\nabla y|^2 dxd\tau + s^2\lambda^2\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}|\varepsilon^{-\frac{1}{2}}\nabla\cdot y|^2 dxd\tau \\ &+ s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|y|^2 dxd\tau \le \widehat{c}_0(\lambda)^2\bigg(s\int_Q e^{2s\alpha}\varphi|F_0|^2 dxd\tau \\ &+ s^{\frac{5}{2}}\lambda^2\int_Q e^{2s\alpha^*}\varphi^{*3}|F_1|^2 dxd\tau + s^4\lambda^4\int_0^T\int_{\mathcal{O}} e^{2s\alpha}\varphi^3|y|^3 dxd\tau\bigg). \end{aligned} \tag{19}$$

We have $\widehat{c}_0(\lambda)\asymp c_0(\lambda)$ and $\widehat{s}_0(\lambda)\asymp \max(s_0(\lambda), c_0(\lambda)^2\lambda^{-8}e^{2\lambda(1-\frac{1}{k})\|\eta\|_\infty})$ for $c_0(\lambda)$, $s_0(\lambda)$ given in Lemma 2.1.

Notice that if we also suppose that

$$\nabla \cdot y_T = 0 \ \text{ in } \ \Omega \quad \text{ and } \quad y_T \cdot n = 0 \ \text{ on } \ \partial\Omega, \tag{20}$$

then with a limiting argument as $\varepsilon \to 0^+$ we deduce a Carleman inequality for the Stokes system:

$$\left\{ \begin{array}{rcll} -\partial_t y - \Delta y + \nabla p &=& f & \text{in } Q, \\ \nabla \cdot y &=& 0 & \text{in } Q, \\ y &=& 0 & \text{on } \Sigma, \\ y(T) &=& y_T & \text{in } \Omega. \end{array} \right. \tag{21}$$

Indeed, suppose first that $y_T \in (H_0^1(\Omega))^d$ satisfies (20). From the uniform analyticity of the semigroup generated by the penalized Stokes operator (see [1]), for ε small enough we have that the solution y_ε to (11) is such that $(y_\varepsilon, \varepsilon^{-1}\nabla \cdot y_\varepsilon)$ is uniformly (in ε) bounded in

$$L^2(0,T;(H^2(\Omega))^d) \cap H^1(0,T;(L^2(\Omega))^d) \times H^1(\Omega)/\mathbb{R}.$$

Then $(y_\varepsilon, \varepsilon^{-1}\nabla \cdot y_\varepsilon)$ converges weakly to (y,p) for such a topology. Moreover, since $\int_0^T \int_{\mathcal{O}} e^{2s\alpha}\varphi^3 |y_\varepsilon|^2 dx d\tau$ is bounded then $(y_\varepsilon, \varepsilon^{-1}\nabla \cdot y_\varepsilon)$ also converges weekly to (y,p) in the sense of the weighted norm involved at the left of the Carleman inequality (16). Thus, passing to the limit in (11) yields that (y,p) satisfies (21). Moreover, from an obvious compactness argument we deduce that $y_\varepsilon \to y \in (L^2(Q))^d$ strongly and then that $\int_0^T \int_{\mathcal{O}} e^{2s\alpha}\varphi^3 |y_\varepsilon|^2 dx d\tau$ converges to $\int_0^T \int_{\mathcal{O}} e^{2s\alpha}\varphi^3 |y|^2 dx d\tau$. Finally, by passing to the limit inf in (16) we obtain the next

Theorem 2.4. *Assume (10), (20), (12), (13) and $k \geq 4$. There exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ there exist two constants $\widehat{c}_0(\lambda) > 1$ and $\widehat{s}_0(\lambda) > 0$ such that for all $s \geq \widehat{s}_0(\lambda)$ the solution to (21) obeys:*

$$\begin{aligned} I_0(s,\lambda,y) + s^{-1} \int_Q e^{2s\widehat{\alpha}} \widehat{\varphi}^{-1} |\nabla p|^2 dx d\tau &\leq \widehat{c}_0(\lambda) I_1(s,\lambda,f) \\ &\quad + \widehat{c}_0(\lambda)^2 s^4 \lambda^4 \int_0^T \int_{\mathcal{O}} e^{2s\alpha}\varphi^3 |y|^2 dx d\tau. \end{aligned} \tag{22}$$

We have $\widehat{c}_0(\lambda) \asymp c_0(\lambda)$ and $\widehat{s}_0(\lambda) \asymp \max(s_0(\lambda), c_0(\lambda)^2 \lambda^{-8} e^{2\lambda(1-\frac{1}{k})\|\eta\|_\infty})$ for $c_0(\lambda)$, $s_0(\lambda)$ given in Lemma 2.1. Note that the implicit constants of $\asymp$ depend on $\|z_{i,j}\|_{L^\infty(0,T;(W^{1,\infty}(\Omega))^d}$, $i = 1, 2$, $j = 1, \ldots, n$.

From Theorem 2.3 we also deduce the following

Theorem 2.5. *Assume (10), (20), (17) and $k \geq 4$. There exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ there exist two constants $\widehat{c}_0(\lambda) > 1$ and $\widehat{s}_0(\lambda) > 0$ such that for all $s \geq \widehat{s}_0(\lambda)$ the solution to (21) obeys:*

$$\begin{aligned} s^2\lambda^2 \int_Q e^{2s\widehat{\alpha}} \widehat{\varphi} |\nabla y|^2 dx d\tau + s^3\lambda^4 \int_Q e^{2s\alpha}\varphi^3 |y|^2 dx d\tau \leq \\ \widehat{c}_0(\lambda)^2 \Big(s \int_Q e^{2s\alpha}\varphi |F_0|^2 dx d\tau + s^{\frac{5}{2}} \lambda^2 \int_Q e^{2s\alpha^*} \varphi^{*3} |F_1|^2 dx d\tau \\ + s^4\lambda^4 \int_0^T \int_{\mathcal{O}} e^{2s\alpha}\varphi^3 |y|^3 dx d\tau \Big). \end{aligned} \tag{23}$$

We have $\widehat{c}_0(\lambda) \asymp c_0(\lambda)$ *and* $\widehat{s}_0(\lambda) \asymp \max(s_0(\lambda), c_0(\lambda)^2\lambda^{-8}e^{2\lambda(1-\frac{1}{k})\|\eta\|_\infty})$ *for* $c_0(\lambda)$, $s_0(\lambda)$ *given in Lemma 2.1.*

Remark 2. Several comments about the above results are in order.
(i) Inequality (16) improves the one given in [22, Thm. 4.1]. Inequality (22) improves the ones of [8, 17], in terms of the powers of s and of the weight functions at the right of the inequality (22). The same comment can be made between (23) and the Carleman inequality for the energy solution of the Stokes system obtained in [18, Lemma 4]. Note the specific powers of s for the terms involving $\nabla \times y$ in (22).
(ii) The dependence in T of the constants is not given in Theorems 2.2. It is due to the fact that such a dependence is not explicit in the Carleman inequality of [22] which is a key tool of the proof of Theorem 2.2. Such a dependence is also a challenging question since it permits to determine the cost of the null control of the penalized Stokes equations, see [9] for details.
(iii) The proof of inequality (22) is much more simple than the one of analogous known inequality. In particular we avoid a local estimation of the pressure, see steps 3 to 7 of [8] or the Appendix of [17]. It allows to obtain a new Carleman inequality for the adjoint system of (6) with no regularity assumptions on z_t but with an extra assumption on ∇z, namely $z \in L^\infty(0,T;(W^{1,\infty}(\Omega))^d)$ instead of $z \in (L^\infty(Q))^d$ and $z_t \in L^2(0,T;(L^\sigma(\Omega))^d)$ with $\sigma > 1$ if $d = 2$ and $\sigma > \frac{6}{5}$ if $d = 3$, see [8, 9]. In fact, it is pointed out in section 3.1 that if $c_0(\lambda)$ in Lemma 2.1 obeys $\lambda^{-2}c_0(\lambda) \to 0$ as $\lambda \to +\infty$ then the Carleman inequality for the Oseen system can be obtained for $z \in (L^\infty(Q))^d$ only.

3. Applications.

3.1. **Oseen and penalized Oseen equations.** The first consequence of Theorems 2.2 is the uniform null controllability of system (1) for $z \in L^\infty(0,T;(W^{1,\infty}(\Omega))^d)$. Indeed, by noticing that $\nabla(z\cdot y) = {}^t(\nabla z)y + {}^t(\nabla y)z$ if we apply (16) with $n = 1$, $f_0 = 2^{-1}(\nabla\cdot z)y - 2^{-1}\,{}^t(\nabla z)y$, $f_1 = 0$, $f_{1,1} = y$, $z_{1,1} = z$, $f_{2,1} = 2^{-1}y$, $z_{2,1} = z$ in (12), then for $\lambda = \lambda_0$ fixed and $s \geq \widehat{s}_0(\lambda_0)$ large enough so that f_0, $(\nabla f_{1,1})z_{1,1}$, $(\nabla f_{2,1})z_{2,1}$ are absorbed in $I_0(s,\lambda_0,y)$, we obtain the following global Carleman estimates for system (4):

$$I_0(s,\lambda_0,y) + s^{-1}\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1}\frac{1}{\varepsilon^2}|\nabla(\nabla\cdot y)|^2 dx d\tau \leq C_0 s^4\int_0^T\int_{\mathcal{O}} e^{2s\alpha}\varphi^3|y|^2 dx d\tau,$$

for $s \geq s_0$ and $s_0 > 0$, $C_0 > 0$ large enough. The uniform null controllability of system (1) can then be deduced from the above inequality, as it is explained in the introduction of [9].

Note also that if $\nabla\cdot z = 0$, an obvious analogous argument applies with Theorem 2.4 to recover the null controllability of the Oseen system (6). Indeed, the use of (22) for the adjoint system of (6) which is given by:

$$\begin{cases} -\partial_t y - \Delta y - (\nabla y)z - {}^t(\nabla y)z + \nabla p = 0 & \text{in } Q, \\ \nabla\cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(T) = y_T & \text{in } \Omega, \end{cases} \tag{24}$$

provides the following estimate for the solution to (24) for $s \geq s_0$ and $s_0 > 0$, $C_0 > 0$ large enough:

$$I_0(s, \lambda_0, y) + s^{-1} \int_Q e^{2s\widehat{\alpha}} \widehat{\varphi}^{-1} |\nabla p|^2 dx d\tau \leq C_0 s^4 \int_0^T \int_{\mathcal{O}} e^{2s\alpha} \varphi^3 |y|^2 dx d\tau.$$

In fact, the above Carleman inequality can be recovered if we only assume $z \in (L^\infty(Q))^d$ and if $c_0(\lambda)\lambda^{-2} \to 0$ as $\lambda \to +\infty$ or if $\|z\|_\infty$ is small enough. Indeed, by applying (22) with $n = 0$, $f_0 = (\nabla y)z + {}^t(\nabla y)z$, $f_1 = 0$, for $\lambda \geq \lambda_0$, $s \geq \widehat{s}_0(\lambda)$ we obtain

$$I_0(s, \lambda, y) + s^{-1} \int_Q e^{2s\widehat{\alpha}} \widehat{\varphi}^{-1} |\nabla p|^2 dx d\tau \leq 2s\widehat{c}_0(\lambda) \|z\|_\infty^2 \int_Q e^{2s\widehat{\alpha}} \widehat{\varphi} |\nabla y|^2 dx d\tau + \widehat{c}_0(\lambda)^2 s^4 \lambda^4 \int_0^T \int_{\mathcal{O}} e^{2s\alpha} \varphi^3 |y|^2 dx d\tau. \tag{25}$$

Then in view of (14) the first term at the right of the inequality can be absorbed for λ and z such that

$$2\lambda^{-2} c_0(\lambda) \|z\|_\infty^2 \leq 1.$$

However, the same argument cannot be applied to recover a uniform Carleman inequality for the penalized system (4) with $z \in (L^\infty(Q))^d$ (eventually small) only. It is due to the presence of the additional terms $(\nabla \cdot z)y$ and $\nabla(z \cdot y)$ which contain first order derivatives of z. We must at least suppose $z \in L^\infty(0, T; (W^{1,r}(\Omega))^d)$ for $r > 2$ if $d = 2$ or $r = 3$ if $d = 3$. Under this assumption, if $c_0(\lambda)\lambda^{-2} \to 0$ as $\lambda \to +\infty$ or if $\|z\|_\infty^2 + \|\nabla \cdot z\|_{L^\infty(L^r)}^2$ is small enough, there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ and for all $s \geq \max(\widehat{s}_0(\lambda), C\widehat{c}_0(\lambda)^2 \lambda^{-4} \|\nabla z\|_{L^\infty(L^r)}^2)$ and $C > 0$ large enough the solution to (4) obeys

$$I_0(s, \lambda, y) + s^{-1} \int_Q e^{2s\widehat{\alpha}} \widehat{\varphi}^{-1} \frac{1}{\varepsilon^2} |\nabla(\nabla \cdot y)|^2 dx d\tau \leq c_0(\lambda)^2 s^4 \lambda^4 \int_0^T \int_{\mathcal{O}} e^{2s\alpha} \varphi^3 |y|^2 dx d\tau.$$

This can be obtained from (16) with $n = 0$, $f_0 = (\nabla y)z + {}^t(\nabla y)z + 2^{-1}(\nabla \cdot z)y$, $f_1 = -2^{-1}\nabla(z \cdot y) = -2^{-1}\, {}^t(\nabla y)z - 2^{-1}\, {}^t(\nabla z)y$ and noting that $\nabla \times f_1 = 0$. Indeed, for such f_0 and f_1 we have

$$s \int_Q e^{2s\alpha} \varphi |f_0|^2 dx d\tau \leq 2s\|z\|_\infty^2 \int_Q e^{2s\alpha} \varphi |\nabla y|^2 dx d\tau + s \int_Q e^{2s\alpha} \varphi |\nabla \cdot z|^2 |y|^2 dx d\tau$$

$$s^{\frac{1}{2}} \int_Q e^{2s\alpha} \varphi |f_1|^2 dx d\tau \leq s^{\frac{1}{2}} \|z\|_\infty^2 \int_Q e^{2s\alpha} \varphi |\nabla y|^2 dx d\tau + s^{\frac{1}{2}} \int_Q e^{2s\alpha} \varphi |\nabla z|^2 |y|^2 dx d\tau$$

and from the continuous embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2r}{r-2}}(\Omega)$ we obtain

$$\begin{aligned} s^{\frac{1}{2}} \int_Q e^{2s\alpha} \varphi |\nabla z|^2 |y|^2 dx d\tau &\leq s^{\frac{1}{2}} \|\nabla z\|_{L^\infty(L^r)}^2 \|e^{s\alpha} \varphi^{\frac{1}{2}} y\|_{L^2(0,T;(H_0^1(\Omega))^d)}^2 \\ &\leq C\|\nabla z\|_{L^\infty(L^r)}^2 \Big(s^{\frac{5}{2}} \lambda^2 \int_Q e^{2s\alpha} \varphi^2 |y|^2 dx d\tau \\ &\qquad + s^{\frac{1}{2}} \int_Q e^{2s\alpha} \varphi |\nabla y|^2 dx d\tau \Big), \end{aligned}$$

and analogously,

$$s\int_Q e^{2s\alpha}\varphi|\nabla\cdot z|^2|y|^2dxd\tau \le C\|\nabla\cdot z\|^2_{L^\infty(L^r)}\Big(s^3\lambda^2\int_Q e^{2s\alpha}\varphi^2|y|^2dxd\tau + s\int_Q e^{2s\alpha}\varphi|\nabla y|^2dxd\tau\Big).$$

Then for $s \ge \max(\widehat{s}_0(\lambda), C\widehat{c}_0(\lambda)^2\lambda^{-4}\|\nabla z\|^2_{L^\infty(L^r)})$ and $C>0$, $C_1>0$ large enough we have

$$\begin{aligned} I_0(s,\lambda,y)+s^{-1}\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1}\frac{1}{\varepsilon^2}|\nabla(\nabla\cdot y)|^2dxd\tau + s\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}|\varepsilon^{-\frac{1}{2}}\nabla\cdot y|^2dxd\tau \\ \le C_1\widehat{c}_0(\lambda)s(\|z\|^2_\infty + \|\nabla\cdot z\|^2_{L^\infty(L^r)})\int_Q e^{2s\alpha}\varphi|\nabla y|^2dxd\tau \\ + \widehat{c}_0(\lambda)^2s^4\lambda^4\int_0^T\int_{\mathcal{O}} e^{2s\alpha}\varphi^3|y|^2dxd\tau, \end{aligned} \tag{26}$$

and in view of (14), the first term at the right of the inequality can be absorbed for λ and z such that

$$\lambda^{-2}c_0(\lambda)(\|z\|^2_\infty + \|\nabla\cdot z\|^2_{L^\infty(L^r)}) < 1/C_1.$$

Finally, let us underline that with (18), we can obtain a nonuniform Carleman inequality for the penalized system (4) with $z\in(L^\infty(Q))^d$ if $c_0(\lambda)\lambda^{-1}\to 0$ as $\lambda\to+\infty$ or $\|z\|_\infty$ small with respect to $\sqrt{\varepsilon}$. Indeed, by remarking that $(\nabla y)z + (\nabla\cdot z)y = \nabla\cdot(y\,{}^tz)$ and applying (18), with $F_0 = 2^{-1}(\nabla y)z + {}^t(\nabla y)z$ and $F_1 = 2^{-1}((y\,{}^tz)-(y\cdot z)I)$, then for $\lambda>\lambda_0$ such that

$$\lambda^{-1}c_0(\lambda)\|z\|_\infty \le \sqrt{\varepsilon}$$

and for $s\ge\widehat{s}_0(\lambda)$ one gets

$$\begin{aligned} \varepsilon s\lambda^2\int_Q e^{2s\alpha}\varphi|\nabla y|^2dxd\tau \quad + \quad s\lambda^2\int_Q e^{2s\alpha}\varphi|\nabla\cdot y|^2dxd\tau + s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|y|^2dxd\tau \\ \le \quad \widehat{c}_0(\lambda)^2s^4\lambda^4\int_0^T\int_{\mathcal{O}} e^{2s\alpha}\varphi^3|y|^3dxd\tau. \end{aligned}$$

3.2. **Micropolar fluid system.** Next, let us show another example of application of Theorems 2.2 and 2.4 which allows to recover in a very simple way the null controllability result obtained in [7] for a micropolar fluid system. According to the above quoted work, when $d=3$ the local controllability to trajectory of some micropolar fluid system is reduced to the observability of a linear system of type

$$\left\{\begin{array}{rcll} -\partial_t y - \Delta y - (\nabla y)(\overline{y}+w) - {}^t(\nabla y)\overline{y} + \nabla p &=& \nabla\times\psi + {}^t(\nabla\psi)\overline{\psi} & \text{in } Q, \\ -\partial_t\psi - \Delta\psi - \nabla(\nabla\cdot\psi) - (\nabla\psi)(\overline{y}+w) &=& \nabla\times y & \text{in } Q, \\ \nabla\cdot y &=& 0 & \text{in } Q, \\ y=\psi &=& 0 & \text{on } \Sigma, \\ (y(T),\psi(T)) &=& (y_T,\psi_T) & \text{in } \Omega, \end{array}\right. \tag{27}$$

where y and ψ represent a velocity and an angular velocity respectively, see [7, eq.(23), p.425]. If we suppose that $\overline{y}$, w and $\overline{\psi}$ belong to $L^\infty(0,T;(W^{1,\infty}(\Omega))^3)$, then (22) with $n=2$, $f_0=\nabla\times\psi$, $f_{1,1}=f_{2,1}=y$, $z_{1,1}=\overline{y}+w$, $z_{2,1}=\overline{y}$, $f_{1,2}=z_{1,2}=0$, $f_{2,2}=\psi$ and $z_{2,2}=\overline{\psi}$ in (12) gives for $s\geq s_0$ and $s_0>0$, $C_0>0$ large enough:

$$I_0(s,\lambda_0,y)+s^{-1}\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1}|\nabla p|^2dxd\tau\leq$$
$$C_0\bigg(\int_Q e^{2s\alpha}(s^{\frac{1}{2}}\varphi|\nabla\psi|^2+s\varphi|\nabla\times\psi|^2)dxd\tau+s^4\int_0^T\int_{\mathcal{O}}e^{2s\alpha}\varphi^3|y|^2dxd\tau\bigg)$$

and (16) with $\varepsilon=1$, and $n=1$, $f_0=\nabla\times y$, $f_{1,1}=\psi$, $z_{1,1}=\overline{y}+w$, $f_{2,1}=z_{2,1}=0$ in (12) gives for $s\geq s_0$ and $s_0>0$, $C_0>0$ large enough:

$$I_0(s,\lambda_0,\psi)+s^{-1}\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1}|\nabla(\nabla\cdot\psi)|^2dxd\tau\leq$$
$$C_0\bigg(s\int_Q e^{2s\alpha}\varphi|\nabla\times y|^2dxd\tau+s^4\int_0^T\int_{\mathcal{O}}e^{2s\alpha}\varphi^3|\psi|^2dxd\tau\bigg).$$

Then by summing the two above inequalities and increasing $s_0>$, $C_0>0$ to absorb the global term at their right, we obtain the following estimate for the solution to (27) for $s\geq s_0$, $s_0>$, $C_0>0$ large enough:

$$I_0(s,\lambda_0,y)+I_0(s,\lambda_0,\psi)+s^{-1}\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1}(|\nabla p|^2+|\nabla(\nabla\cdot\psi)|^2)dxd\tau\leq$$
$$C_0s^4\int_0^T\int_{\mathcal{O}}e^{2s\alpha}\varphi^3(|y|^2+|\psi|^2)dxd\tau.$$

Note that on the contrary to [7], here no time regularity is required for $\overline{y}$, $\overline{\psi}$, w. In the above quoted work, $\overline{y}$, $\overline{\psi}$ are vector fields of $(L^\infty(Q))^3\cap H^1(0,T;(L^2(\Omega))^3)$ such that $\overline{y}$, $\overline{\psi}$ belong to $L^2(0,T;(H^2(\Omega))^3)$ and w belongs to $L^\infty(0,T;(H^\kappa(\Omega))^3)$ for $\kappa>\frac{1}{2}$, see [7, eq. (21), p.424]. Note also that if $\overline{y}$, w and $\overline{\psi}$ are only in $L^\infty(Q)$, then if $c_0(\lambda)\lambda^{-2}\to 0$ as $\lambda\to+\infty$ or if $\|\overline{y}\|_\infty+\|w\|_\infty+\|\overline{\psi}\|_\infty$ is small enough, then by proceeding as for (25) we can recover the above Carleman estimate.

Finally, on shall also underline that an analogous argument can be used to obtain in a simple way a Carleman inequality for the linearized adjoint Boussinesq system, as in [16, Prop. 1].

3.3. **Controllability to trajectories of Navier-Stokes and penalized Navier-Stokes equations.** First, let us prove the local controllability to trajectories of the penalized Navier-Stokes equations. Namely, for

$$z_0,\ \overline{z}_0\in(L^2(\Omega))^d,\tag{28}$$

and given a solution $z\in L^\infty(0,T;(W^{1,\infty}(\Omega))^d)$ to:

$$\begin{cases}\partial_t z-\Delta z+B(z,z)-\dfrac{1}{\varepsilon}\nabla(\nabla\cdot z) &= 0 \quad \text{in } Q,\\ z &= 0 \quad \text{on } \Sigma,\\ z(0) &= z_0 \quad \text{in } \Omega,\end{cases}\tag{29}$$

we look for a pair $(\overline{z}, h)$ solution to

$$\left\{\begin{array}{rcll} \partial_t\overline{z} - \Delta\overline{z} + B(\overline{z},\overline{z}) - \dfrac{1}{\varepsilon}\nabla(\nabla\cdot\overline{z}) & = & h\mathbf{1}_{\mathcal{O}} & \text{in } Q, \\ \overline{z} & = & 0 & \text{on } \Sigma, \\ \overline{z}(0) = \overline{z}_0 \quad \text{and} \quad \overline{z}(T) & = & z(T) & \text{in } \Omega. \end{array}\right. \tag{30}$$

Note that if one sets $v = \overline{z} - z$ and $v_0 = \overline{z}_0 - z_0$ then it suffices to find a pair (v, h) solution to:

$$\left\{\begin{array}{rcll} \partial_t v - \Delta v + B(z,v) + B(v,z) + B(v,v) - \dfrac{1}{\varepsilon}\nabla(\nabla\cdot v) & = & h\mathbf{1}_{\mathcal{O}} & \text{in } Q, \\ v & = & 0 & \text{on } \Sigma, \\ v(0) = v_0 \quad \text{and} \quad v(T) & = & 0 & \text{in } \Omega. \end{array}\right. \tag{31}$$

Following the strategy of [8] we need to introduce new weight functions. Set

$$\beta(x,t) = \frac{e^{\lambda\eta(x)} - e^{2\lambda\|\eta\|_\infty}}{\tilde{\ell}(t)^k}, \quad \gamma(x,t) = \frac{e^{\lambda(\eta(x)+\|\eta\|_\infty)}}{\tilde{\ell}(t)^k}$$

$$\text{where} \quad \tilde{\ell}(t) = \left\{\begin{array}{ll} \dfrac{T}{2} & \text{if } t \in \left[0, \dfrac{T}{2}\right] \\ T - t & \text{if } t \in \left(\dfrac{T}{2}, T\right], \end{array}\right.$$

and

$$\widehat{\beta}(x,t) = \min_{x\in\Omega}\beta(x,t) = \frac{1 - e^{2\lambda\|\eta\|_\infty}}{\tilde{\ell}(t)^k}, \qquad \widehat{\gamma}(t) = \min_{x\in\Omega}\gamma(x,t) = \frac{e^{\lambda\|\eta\|_\infty}}{\tilde{\ell}(t)^k}.$$

Moreover, for $r \in [0,1]$ we introduce the following weighted spaces:

$$\begin{aligned} F_r &\overset{\text{def}}{=} L^2(e^{-s\widehat{\beta}}\widehat{\gamma}^{-\frac{1}{2}}(0,T); (H^{r-1}(\Omega))^d), \\ W_r &\overset{\text{def}}{=} L^2(e^{-s\frac{\widehat{\beta}}{2}}\widehat{\gamma}^{-\frac{1}{4}}(0,T); (H^{1+r}(\Omega))^d) \cap (L^2(e^{-s\beta}\gamma^{-\frac{1}{2}}Q))^d \\ &\quad \cap C(e^{-s\frac{\widehat{\beta}}{2}}\widehat{\gamma}^{-\frac{1}{4}}[0,T]; (H^r(\Omega))^d), \\ E_r &\overset{\text{def}}{=} W_r \times (L^2(e^{-s\beta}\gamma^{-\frac{3}{2}}(0,T)\times\mathcal{O}))^d. \end{aligned}$$

as well as the following norm on $(H^r_0(\Omega))^d$:

$$\|v\|^2_{r,\varepsilon} \overset{\text{def}}{=} \|v\|^2_{(H^r_0(\Omega))^d} + \varepsilon^{-\frac{r}{2}}\|\nabla\cdot v\|^2_{(H^{r-1}(\Omega))^d}. \tag{32}$$

For $(f, v_0) \in F_r \times (H^r_0(\Omega))^d$ we first prove the existence of a pair (v,h) solution to the nonhomogeneous problem

$$\left\{\begin{array}{rcll} \partial_t v - \Delta v + B(z,v) + B(v,z) - \dfrac{1}{\varepsilon}\nabla(\nabla\cdot v) & = & f + h\mathbf{1}_{\mathcal{O}} & \text{in } Q, \\ v & = & 0 & \text{on } \Sigma, \\ v(0) = v_0, \qquad v(T) & = & 0 & \text{in } \Omega. \end{array}\right. \tag{33}$$

Theorem 3.1. *Let* $z \in L^\infty(0,T;(W^{1,\infty}(\Omega))^d)$*, let* $r \in [0,1]$ *and let* $(f,v_0) \in F_r \times (H_0^r(\Omega))^d$*. Then* (33) *admits a solution* $(v,h) \in E_r$ *such that* $\|(v,h)\|_{E_r} \leq C_1(\|f\|_{F_r} + \|v_0\|_{r,\varepsilon})$*, for a constant* $C_1 > 0$ *independent on* ε*.*

Thus, with a fixed point argument we obtain a solution of (31) for v_0 small in $(H_0^r(\Omega))^d$ in the sense of (32). It yields the following

Theorem 3.2. *Assume* (28)*, let* $z \in L^\infty(0,T;(W^{1,\infty}(\Omega))^d)$ *be a solution to* (29)*, let* $r \in [\frac{d-2}{2},1]$*,* $r \neq 0$*, and* $\overline{z}_0 \in \{z_0\} + (H_0^r(\Omega))^d$*. There exist* $\rho > 0$ *and* $\mu > 0$ *such that, if* $\delta \in (0,\mu)$ *and* $\|z_0 - \overline{z}_0\|_{r,\varepsilon} < \delta$*, then there exists a pair* $(\overline{z},h) \in \{(z,0)\} + E_r$ *solution to* (30) *such that* $\|(\overline{z}-z,h)\|_{E_r} \leq \rho\delta$*. Moreover,* ρ *and* μ *do not depend on* ε*.*

Suppose now that

$$z_0,\ \overline{z}_0 \in V_n^0(\Omega) \stackrel{\text{def}}{=} \{v \in (L^2(\Omega))^d \mid \ \nabla\cdot v = 0 \text{ in } \Omega,\ v\cdot n = 0 \text{ on } \partial\Omega\}, \tag{34}$$

and that $(z,\pi) \in L^\infty(0,T;(W^{1,\infty}(\Omega))^d) \times L^2(Q)$ is a solution to the Navier-Stokes equations:

$$\begin{cases} \partial_t z - \Delta z + (z\cdot\nabla)z + \nabla\pi = 0 & \text{in } Q, \\ \nabla\cdot z = 0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = z_0 & \text{in } \Omega. \end{cases} \tag{35}$$

Then by combining Theorem 3.2 with a limiting argument as $\varepsilon \to 0^+$, we obtain a triplet $(\overline{z},\overline{\pi},h)$ solution to

$$\begin{cases} \partial_t \overline{z} - \Delta \overline{z} + (\overline{z}\cdot\nabla)\overline{z} + \nabla\overline{\pi} = h\mathbf{1}_{\mathcal{O}} & \text{in } Q, \\ \nabla\cdot \overline{z} = 0 & \text{in } Q, \\ \overline{z} = 0 & \text{on } \Sigma, \\ \overline{z}(0) = \overline{z}_0 \quad \text{and} \quad \overline{z}(T) = z(T) & \text{in } \Omega. \end{cases} \tag{36}$$

Theorem 3.3. *Assume* (34)*, let* $z \in L^\infty(0,T;(W^{1,\infty}(\Omega))^d)$ *be a solution to* (35)*, let* $r \in [\frac{d-2}{2},1]$*,* $r \neq 0$*, and* $\overline{z}_0 \in \{z_0\} + (H_0^r(\Omega))^d \cap V_n^0(\Omega)$*. There exist* $\rho > 0$ *and* $\mu > 0$ *such that, if* $\delta \in (0,\mu)$ *and* $\|z_0 - \overline{z}_0\|_{(H^r(\Omega))^d} < \delta$*, then there exists a pair* $(\overline{z},h,\overline{\pi}) \in \{(z,0,\pi)\} + E_r \times H^r(\Omega)$ *solution to* (36) *such that* $\|(\overline{z}-z,h)\|_{E_r} \leq \rho\delta$*.*

4. **Some technical results.** First, let us recall some properties satisfied by the weight functions (7), (8) and (9). We have:

$$T^{-1}|\ell(t)| + |(\ell)_t(t)| + T|(\ell)_{tt}(t)| \leq c, \tag{37}$$

and for all $\lambda > 1$, $i = 1, \dots, d$ and $b \geq a$:

$$\begin{aligned}
\lambda^{-2}|\partial^2_{x_i}\varphi(x,t)| + \lambda^{-1}|\partial_{x_i}\varphi(x,t)| &\leq c\varphi(x,t), && (38)\\
|\varphi^a(x,t)| &\leq cT^{(b-a)k}\varphi^b(x,t), && (39)\\
|\widehat{\varphi}^a(t)| &\leq cT^{(b-a)k}\widehat{\varphi}^b(t), && (40)\\
|(\widehat{\varphi})_t(t)| + e^{(\frac{1}{k}-1)\lambda\|\eta\|_\infty}|(\widehat{\alpha})_t(t)| &\leq c\widehat{\varphi}^{1+\frac{1}{k}}(t), && (41)\\
|(\varphi)_t(x,t)| + |(\alpha)_t(x,t)| &\leq cT^{k-1}\varphi^2(x,t), && (42)\\
|(\alpha)_{tt}(x,t)| &\leq cT^{2k-2}\varphi^3(x,t), && (43)
\end{aligned}$$

for some constant $c > 0$ which does not depend on T or λ.

In the following C denotes a generic positive constant that may change from line to line, which may depends on T, k, $\|z_{i,j}\|_{L^\infty(0,T;(W^{1,\infty}(\Omega))^d)}$ and on the geometry, but which is independent on s, λ and ε.

Next, we state two global Carleman inequalities for parabolic equations. The first one is obtained from [9, Lemma 1.3].

Lemma 4.1. *Let $T > 0$ and $\delta \in [0,1]$. There exist three constants $C_0 > 0$, $C_1 > 0$ and $C_2 > 0$ independent on T and δ, such that for all $\lambda \geq C_1$ and for all $s \geq C_2(\delta T^{k-1} + T^k)$, the following inequality holds:*

$$\begin{aligned}
s^{-1}\int_Q e^{2s\alpha}\varphi^{-1}(\delta^2|\partial_t q|^2 + |\Delta q|^2)dxd\tau + s\lambda^2\int_Q e^{2s\alpha}\varphi|\nabla q|^2 dxd\tau \\
+ s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|q|^2dxd\tau \leq C_0\Big(\int_Q e^{2s\alpha}|\delta\partial_t q + \Delta q|^2 dxd\tau \\
+ s^3\lambda^4\int_0^T\int_\omega e^{2s\alpha}\varphi^3|q|^2dxd\tau\Big),
\end{aligned}$$

for all $q \in C^2(\overline{Q})$ with $q = 0$ on Σ.

Proof. The proof of the result for $\delta = 1$ can be found in [9, Lemma 1.3]. The case $\delta \in (0,1)$ is then deduced with a change of variable argument. Indeed, let apply the results for $\delta = 1$ and for a time horizon T/δ to the function $\tilde{q} \in C^2([0,T/\delta] \times \overline{\Omega})$ defined by $\tilde{q}(\tau,x) = q(\delta\tau,x)$ for $(\tau,x) \in [0,T/\delta]\times\Omega$. The weight functions are then given by $\tilde{\varphi}(\tau,x) = \delta^k\varphi(\delta\tau,x)$ and $\tilde{\alpha}(\tau,x) = \delta^k\alpha(\delta\tau,x)$, where α, φ are defined in (8), and [9, Lemma 1.3] states the existence of three constants $C_0 > 0$, $C_1 > 0$ and $C_2 > 0$ independent on T, δ (but depending on Ω, ω) such that for all $\lambda \geq C_1$ and for all $\tilde{s} \geq C_2((T/\delta)^{k-1} + (T/\delta)^k)$ we have

$$\begin{aligned}
\tilde{s}^{-1}\int_0^{T/\delta}\int_\Omega e^{2\tilde{s}\tilde{\alpha}}\tilde{\varphi}^{-1}(|\partial_t\tilde{q}|^2 + |\Delta\tilde{q}|^2)dxd\tau + \tilde{s}\lambda^2\int_0^{T/\delta}\int_\Omega e^{2\tilde{s}\tilde{\alpha}}\tilde{\varphi}|\nabla q|^2dxd\tau \\
+ \tilde{s}^3\lambda^4\int_0^{T/\delta}\int_\Omega e^{2\tilde{s}\tilde{\alpha}}\tilde{\varphi}^3|\tilde{q}|^2dxd\tau \leq C_0\Big(\int_0^{T/\delta}\int_\Omega e^{2\tilde{s}\tilde{\alpha}}|\partial_t\tilde{q} + \Delta\tilde{q}|^2dxd\tau \\
+ \tilde{s}^3\lambda^4\int_0^{T/\delta}\int_\omega e^{2\tilde{s}\tilde{\alpha}}\tilde{\varphi}^3|q|^2dxd\tau\Big).
\end{aligned}$$

By multiplying the above inequality by δ and noticing that $\partial_t\tilde{q}(\tau,x) = \delta\partial_t q(\delta\tau,x)$, w ith a change of variable $\tau \mapsto \tau/\delta$ one deduces that for $\lambda \geq C_1$ and for $s = \delta^k\tilde{s} \geq$

$C_2(\delta T^{k-1}+T^k)$ we have

$$s^{-1}\int_0^T\int_\Omega e^{2s\alpha}\varphi^{-1}(\delta^2|\partial_t q|^2+|\Delta q|^2)dxd\tau+s\lambda^2\int_0^T\int_\Omega e^{2s\alpha}\varphi|\nabla q|^2dxd\tau$$
$$+s^3\lambda^4\int_0^T\int_\Omega e^{2s\alpha}\varphi^3|q|^2dxd\tau\le C_0\Big(\int_0^T\int_\Omega e^{2s\alpha}|\delta\partial_t q+\Delta q|^2dxd\tau$$
$$+s^3\lambda^4\int_0^{T/\delta}\int_\omega e^{2s\alpha}\varphi^3|q|^2dxd\tau\Big),$$

and the desired result is obtained. □

Remark 3. To be precise, the Carleman inequality for the heat equation provided in [9, Lemma 1.3] is enounced for analogous weight functions corresponding to the case $k=2$. However, since its proof relies on the bounds (38), (39), (42) and (43), one can follow the steps of the proof of [9, Lemma 1.3] to obtain Lemma 4.1 for $\delta=1$.

In the following we will need a slightly different version of Lemma 2.1.

Lemma 4.2. *Let $F_0\in L^2(Q)$ and $F_1\in(L^2(Q))^d$ and let $\psi\in H^{\frac12,1}(Q)$ satisfying*

$$-\partial_t\psi-\Delta\psi=\nabla\cdot F_1+F_0\quad\text{in } Q.$$

There exists $\lambda_0>0$ such that for all $\lambda\ge\lambda_0$ there exist two constants $\widehat{c}_0(\lambda)>1$ and $\widehat{s}_0(\lambda)>0$ and such that for all $s\ge\widehat{s}_0(\lambda)$, the following inequality holds:

$$\int_Q e^{2s\alpha}|\nabla\psi|^2dxd\tau+s^2\lambda^2\int_Q e^{2s\alpha}\varphi^2|\psi|^2dxd\tau\le$$
$$\widehat{c}_0(\lambda)\Big(s^{-1}\lambda^{-2}\int_Q e^{2s\alpha}\varphi^{-1}|F_0|^2dxd\tau+s\int_Q e^{2s\alpha}\varphi|F_1|^2dxd\tau$$
$$+s^2\lambda^2\int_0^T\int_\omega e^{2s\alpha}\varphi|\psi|^2dxd\tau+s^{\frac12}\|\widehat{\varphi}^{\frac14}\psi e^{s\widehat{\alpha}}\|^2_{H^{\frac14,\frac12}(\Sigma)}\Big).$$

Moreover, we have $\widehat{c}_0(\lambda)\asymp c_0(\lambda)$ and $\widehat{s}_0(\lambda)\asymp\max(s_0(\lambda),c_0(\lambda))$ for $c_0(\lambda)$, $s_0(\lambda)$ given in Lemma 2.1.

Proof. First, an easy calculation shows that $\tilde\psi=\varphi^{\frac12}\psi$ satisfies in Q:

$$-\partial_t\tilde\psi-\Delta\tilde\psi=\varphi^{\frac12}F_0-F_1\cdot\nabla(\varphi^{\frac12})+\nabla\cdot(\varphi^{\frac12}F_1)+\Delta(\varphi^{\frac12})\psi-2\nabla\cdot((\nabla\varphi^{\frac12})\psi)-(\varphi^{\frac12})_t\psi.$$

Thus, since from (38) and (42) we deduce that

$$|\nabla(\varphi^{\frac12})|^2\le C\lambda^2\varphi,\quad|\Delta\varphi^{\frac12}|^2\le C\lambda^4\varphi\quad\text{and}\quad|(\varphi^{\frac12})_t|^2\le C\varphi^3,\tag{44}$$

then by applying Lemma 2.1 and taking (39) into account we obtain for s large enough:

$$\int_Q e^{2s\alpha}\varphi^{-1}|\nabla(\varphi^{\frac12}\psi)|^2dxd\tau+s^2\lambda^2\int_Q e^{2s\alpha}\varphi^2|\psi|^2dxd\tau\le$$
$$c_0(\lambda)C\Big(s^{-1}\lambda^{-2}\int_Q e^{2s\alpha}\varphi^{-1}|F_0|^2dxd\tau+s\int_Q e^{2s\alpha}\varphi|F_1|^2dxd\tau$$
$$+s\lambda^2\int_Q e^{2s\alpha}\varphi^2|\psi|^2dxd\tau+s^2\lambda^2\int_0^T\int_\omega e^{2s\alpha}\varphi^2|\psi|^2dxd\tau$$
$$+s^{\frac12}\|\widehat{\varphi}^{\frac14}\psi e^{s\widehat{\alpha}}\|^2_{H^{\frac14,\frac12}(\Sigma)}\Big).$$

Thus for $s \geq C_1 \max(s_0(\lambda), c_0(\lambda))$ with $C_1 > 0$ large enough we can drop the term $s\lambda^2 \int_Q e^{2s\alpha}\varphi^2|\psi|^2 dxd\tau$ at the right of the above inequality, and finally, since by (39) and (44) we have $2\varphi^{-1}|\nabla(\varphi^{\frac{1}{2}}\psi)|^2 = 2\varphi^{-1}|\varphi^{\frac{1}{2}}\nabla\psi + \nabla(\varphi^{\frac{1}{2}})\psi|^2 \geq |\nabla\psi|^2 - C\lambda^2\varphi^2|\psi|^2$, we can conclude by choosing again C_1 large enough. □

5. **Proof of Theorem 2.2.** The proof of the Theorem is divided in 7 steps.

Step 1.

First, we set $\psi = \nabla \times y$ and using in (11) the equality:

$$\nabla(\nabla \cdot y) = \Delta y + \nabla \times (\nabla \times y)$$

we deduce that:

$$-\frac{\varepsilon}{1+\varepsilon}\partial_t y - \Delta y = \frac{\varepsilon}{1+\varepsilon} f + \frac{1}{1+\varepsilon}\nabla \times \psi,$$

where f is given by (12). Then by Lemma 4.1 with $\delta = \varepsilon(1+\varepsilon)^{-1}$ we obtain for $\lambda \geq \lambda_0$, $s \geq s_0$ and $s_0 > 0$, $\lambda_0 > 0$, $C > 0$ large enough:

$$s^{-1}\int_Q e^{2s\alpha}\varphi^{-1}|\Delta y|^2 dxd\tau + s\lambda^2\int_Q e^{2s\alpha}\varphi|\nabla y|^2 dxd\tau + s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|y|^2 dxd\tau \quad \leq$$
$$C\left(\varepsilon^2\int_Q e^{2s\alpha}|f|^2 dxd\tau + \int_Q e^{2s\alpha}|\nabla\psi|^2 dxd\tau + s^3\lambda^4\int_0^T\int_\omega e^{2s\alpha}\varphi^3|y|^2 dxd\tau\right). \quad (45)$$

Step 2.

By applying the curl operator to (11) and recalling (12) we deduce that $\psi = \nabla \times y$ verifies

$$-\partial_t\psi - \Delta\psi = \nabla \times f_0 + \nabla \times f_1 + \sum_{j=1}^{n} \nabla \times \left((\nabla f_{1,j})z_{1,j} + {}^t(\nabla f_{2,j})z_{2,j}\right).$$

Thus, we make the computations:

$$\begin{aligned}
\nabla \times ((\nabla f_{1,j})z_{1,j} + {}^t(\nabla f_{2,j})z_{2,j}) \;&=\; \nabla \times ((\nabla f_{1,j})z_{1,j} + (\nabla f_{2,j})z_{2,j} \\
&+\; ({}^t(\nabla f_{2,j}) - (\nabla f_{2,j}))z_{2,j}) \\
&=\; (\nabla(\nabla \times f_{1,j}))z_{1,j} + (\nabla(\nabla \times f_{2,j}))z_{2,j} \\
&+\; \mathcal{A}(f_{1,j}, z_{1,j}) + \mathcal{A}(f_{2,j}, z_{2,j}) \\
&+\; \nabla \times (\mathcal{B}(z_{2,j})(\nabla \times f_{2,j})), \\
&=\; \nabla \cdot ((\nabla \times f_{1,j}) \otimes z_{1,j}) \\
&+\; \nabla \cdot ((\nabla \times f_{2,j}) \otimes z_{2,j}) \\
&-\; (\nabla \times f_{1,j})(\nabla \cdot z_{1,j}) - (\nabla \times f_{2,j})(\nabla \cdot z_{2,j}) \\
&+\; \mathcal{A}(f_{1,j}, z_{1,j}) + \mathcal{A}(f_{2,j}, z_{2,j}) \\
&+\; \nabla \times (\mathcal{B}(z_{2,j})\nabla \times f_{2,j}),
\end{aligned}$$

where we have used the notations:

$$\begin{cases}
\nabla \cdot (v \otimes z) &= \displaystyle\sum_{j=1}^{2} \partial_{x_j}(vz_j), & & \text{if } d = 2, \\
(\nabla \cdot (w \otimes z))_i &= \displaystyle\sum_{j=1}^{3} \partial_{x_j}(w_i z_j), & i = 1, 2, 3, & \text{if } d = 3,
\end{cases}$$

and

$$\mathcal{A}(w, z) = \nabla w_2 \partial_{x_1} z - \nabla w_1 \partial_{x_2} z, \qquad \mathcal{B}(w) = \begin{pmatrix} w_2 \\ -w_1 \end{pmatrix} \quad \text{if } d = 2,$$

and

$$\mathcal{A}(w,z) = \begin{pmatrix} \nabla w_3 \partial_{x_2} z - \nabla w_2 \partial_{x_3} z \\ \nabla w_1 \partial_{x_3} z - \nabla w_3 \partial_{x_1} z \\ \nabla w_2 \partial_{x_1} z - \nabla w_1 \partial_{x_2} z \end{pmatrix}, \qquad \mathcal{B}(w) = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}$$

if $d = 3$. In the above setting, w_i, z_i, denote the i-th component of w and z respectively. Then we deduce that

$$\begin{aligned} \partial_t \psi - \Delta \psi &= \nabla \times f_0 + \nabla \times f_1 + \sum_{j=1}^{n} \nabla \times (\mathcal{B}(z_{2,j})(\nabla \times f_{2,j})) \\ &+ \sum_{j=1}^{n} \sum_{i=1}^{2} \left(\nabla \cdot ((\nabla \times f_{i,j}) \otimes z_{i,j}) + \mathcal{A}(f_{i,j}, z_{i,j}) - (\nabla \times f_{i,j})(\nabla \cdot z_{i,j}) \right), \end{aligned}$$

and by Lemma 4.2, for $\lambda \geq \lambda_0$, $s \geq c_1 \max(s_0(\lambda), c_0(\lambda))$ and $c_1 > 0$, $\lambda_0 > 0$, $C > 0$ large enough we deduce that:

$$\begin{aligned} &\int_Q e^{2s\alpha} |\nabla \psi|^2 dx d\tau + s^2 \lambda^2 \int_Q e^{2s\alpha} \varphi^2 |\psi|^2 dx d\tau \leq \\ &c_0(\lambda) C \Big(+ s \int_Q e^{2s\alpha} \varphi |f_0|^2 dx d\tau + s^{-1} \lambda^{-2} \int_Q e^{2s\alpha} \varphi^{-1} |\nabla \times f_1|^2 dx d\tau \\ &\quad + \sum_{j=1}^{n} \sum_{i=1}^{2} (s^{-1} \lambda^{-2} \int_Q e^{2s\alpha} \varphi^{-1} |\nabla f_{i,j}|^2 dx d\tau + s \int_Q \varphi |\nabla \times f_{i,j}|^2 dx d\tau) \\ &\quad s^2 \lambda^2 \int_0^T \int_\omega e^{2s\alpha} \varphi |\psi|^2 dx d\tau + s^{\frac{1}{2}} \| \widehat{\varphi}^{\frac{1}{4}} \psi e^{s\widehat{\alpha}} \|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} \Big), \end{aligned}$$

and using the fact that $\varphi^{-1} \leq C\varphi$ and for $\lambda > 0$, $s > 0$ and $C > 0$ large enough we obtain:

$$\begin{aligned} &\int_Q e^{2s\alpha} |\nabla \psi|^2 dx d\tau + s^2 \lambda^2 \int_Q e^{2s\alpha} \varphi^2 |\psi|^2 dx d\tau \leq \\ &c_0(\lambda) C \Big(s^2 \lambda^2 \int_0^T \int_\omega e^{2s\alpha} \varphi |\psi|^2 dx d\tau + I_1(s, \lambda, f) + s^{\frac{1}{2}} \| \widehat{\varphi}^{\frac{1}{4}} \psi e^{s\widehat{\alpha}} \|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} \Big). \end{aligned} \tag{46}$$

Step 3.

Let us bound the boundary term $s^{\frac{1}{2}} \| \widehat{\varphi}^{\frac{1}{4}} \psi e^{s\widehat{\alpha}} \|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)}$. We set

$$w = \widehat{\varphi}^{\frac{1}{4}} y e^{s\widehat{\alpha}}. \tag{47}$$

By recalling that $\psi = \nabla \times y$, and by using the fact that $\widehat{\alpha}$ and $\widehat{\varphi}$ do not depend on the space variable, we deduce that:

$$\| \widehat{\varphi}^{\frac{1}{4}} y e^{s\widehat{\alpha}} \|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} = \| \nabla \times w \|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} \leq \| \nabla w \|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} = \| \frac{dw}{dn} \|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)}, \tag{48}$$

where the last equality is justified by the fact that $w = 0$ on $\partial\Omega$. Thus, since by [15, Thm. 7.2] we know that

$$\| \frac{dw}{dn} \|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} \leq C \| w \|^2_{H^{1,2}(Q)},$$

by combining this with (48) we finally obtain for s large enough:

$$s^{\frac{1}{2}} \| \widehat{\varphi}^{\frac{1}{4}} \psi e^{s\widehat{\alpha}} \|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} \leq C s^{\frac{1}{2}} \| w \|^2_{H^{1,2}(Q)}. \tag{49}$$

Step 4.
For f given by (12) we verify that w defined by (47) obeys:

$$\partial_t w - \Delta w - \frac{1}{\varepsilon}\nabla(\nabla\cdot w) = \widehat{\varphi}^{\frac{1}{4}}e^{s\widehat{\alpha}}f + 4^{-1}\widehat{\varphi}^{-\frac{3}{4}}(\widehat{\varphi})_t e^{s\widehat{\alpha}}y + s(\widehat{\alpha})_t\widehat{\varphi}^{\frac{1}{4}}e^{s\widehat{\alpha}}y,$$

and by successively applying uniform (in ε) regularity results for the instationary penalized Stokes system (see for instance [1]) and (41), we obtain:

$$\|w\|^2_{H^{1,2}(Q)} \le C(\|\widehat{\varphi}^{\frac{1}{4}}e^{s\widehat{\alpha}}f\|^2_{L^2(Q)} + \|\widehat{\varphi}^{\frac{1}{4}+\frac{1}{k}}ye^{s\widehat{\alpha}}\|^2_{L^2(Q)} + s^2e^{\lambda(1-\frac{1}{k})\|\eta\|_\infty}\|\widehat{\varphi}^{\frac{5}{4}+\frac{1}{k}}e^{s\widehat{\alpha}}y\|^2_{L^2(Q)}).$$

Then if we choose $k \ge 4$ we can deduce from (40) that:

$$\|w\|^2_{H^{1,2}(Q)} \le C\Big(\int_Q \widehat{\varphi}e^{2s\widehat{\alpha}}|f|^2dxdt + \int_Q \widehat{\varphi}^3e^{2s\widehat{\alpha}}|y|^2dxdt + s^2e^{\lambda(1-\frac{1}{k})\|\eta\|_\infty}\int_Q \widehat{\varphi}^3e^{2s\widehat{\alpha}}|y|^2dxdt\Big).$$

Finally, combining the above inequality with (49) gives for $s > 0$ large enough:

$$s^{\frac{1}{2}}\|\widehat{\varphi}^{\frac{1}{4}}\psi e^{s\widehat{\alpha}}\|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} \le C(s^{\frac{1}{2}}\int_Q \widehat{\varphi}e^{2s\widehat{\alpha}}|f|^2dxdt + s^{\frac{5}{2}}e^{\lambda(1-\frac{1}{k})\|\eta\|_\infty}\int_Q \widehat{\varphi}^3e^{2s\widehat{\alpha}}|y|^2dxdt). \tag{50}$$

Step 5.
By combining (50), (46) and (12) we obtain for s large enough:

$$\int_Q e^{2s\alpha}|\nabla\psi|^2dxd\tau + s^2\lambda^2\int_Q e^{2s\alpha}\varphi^2|\psi|^2dxd\tau \le c_0(\lambda)C\Big(I_1(s,\lambda,f) + s^{\frac{5}{2}}e^{\lambda(1-\frac{1}{k})\|\eta\|_\infty}\int_Q e^{2s\alpha}\varphi^3|y|^2dxd\tau + s^2\lambda^2\int_0^T\int_\omega e^{2s\alpha}\varphi|\psi|^2dxd\tau\Big),$$

and with (45) and (12), for $\lambda \ge \lambda_0$, $s \ge c_1\max(s_0(\lambda), c_0(\lambda)^2\lambda^{-8}e^{2\lambda(1-\frac{1}{k})\|\eta\|_\infty})$ and $c_1 > 0$, $\lambda_0 > 0$, $C > 0$ large enough we obtain:

$$\begin{aligned} &s^{-1}\int_Q e^{2s\alpha}\varphi^{-1}|\Delta y|^2dxd\tau + \int_Q e^{2s\alpha}|\nabla(\nabla\times y)|^2dxd\tau \\ &+ s\lambda^2\int_Q e^{2s\alpha}\varphi|\nabla y|^2dxd\tau + s^2\lambda^2\int_Q e^{2s\alpha}\varphi^2|\nabla\times y|^2dxd\tau \\ &+ s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|y|^2dxd\tau \le c_0(\lambda)C\Big(I_1(s,\lambda,f) \\ &+ s^2\lambda^2\int_0^T\int_\omega e^{2s\alpha}\varphi|\nabla\times y|^2dxd\tau + s^3\lambda^4\int_0^T\int_\omega e^{2s\alpha}\varphi^3|y|^2dxd\tau\Big). \end{aligned} \tag{51}$$

Step 6.
Here we use a now classical localization argument to drop the second term at the right of inequality (51). We recall that $\overline{\omega} \subset \mathcal{O}$ and let $\theta \in C_c^\infty(\mathcal{O})$, $0 \leq \theta \leq 1$ and $\theta \equiv 1$ in ω. First, we have:

$$s^2 \int_0^T \int_\omega e^{2s\alpha}\varphi|\nabla \times y|^2 dx d\tau \leq s^2 \int_0^T \int_{\mathcal{O}} \theta e^{2s\alpha}\varphi|\nabla \times y|^2 dx d\tau,$$

and with the following computations if $d = 2$:

$$\begin{aligned} s^2 \int_0^T \int_{\mathcal{O}} \theta e^{2s\alpha}\varphi|\nabla \times y|^2 dx d\tau = {} & s^2 \int_0^T \int_{\mathcal{O}} \theta e^{2s\alpha}\varphi \nabla \times (\nabla \times y) \cdot y dx d\tau \\ & + s^2 \int_0^T \int_{\mathcal{O}} e^{2s\alpha}\varphi((\nabla \times \theta)(\nabla \times y)) \cdot y dx d\tau \\ & + \lambda s^2 \int_0^T \int_{\mathcal{O}} \theta e^{2s\alpha}\varphi((\nabla \times \eta)(\nabla \times y)) \cdot y dx d\tau \\ & + 2e^{-\lambda\|\eta\|_\infty}\lambda s^3 \int_0^T \int_{\mathcal{O}} \theta e^{2s\alpha}\varphi^2((\nabla \times \eta)(\nabla \times y)) \cdot y dx d\tau, \end{aligned}$$

or with the following computations if $d = 3$:

$$\begin{aligned} s^2 \int_0^T \int_{\mathcal{O}} \theta e^{2s\alpha}\varphi|\nabla \times y|^2 dx d\tau = {} & s^2 \int_0^T \int_{\mathcal{O}} \theta e^{2s\alpha}\varphi \nabla \times (\nabla \times y) \cdot y dx d\tau \\ & + s^2 \int_0^T \int_{\mathcal{O}} e^{2s\alpha}\varphi(\nabla\theta \times (\nabla \times y)) \cdot y dx d\tau \\ & + \lambda s^2 \int_0^T \int_{\mathcal{O}} \theta e^{2s\alpha}\varphi(\nabla\eta \times (\nabla \times y)) \cdot y dx d\tau \\ & + 2e^{-\lambda\|\eta\|_\infty}\lambda s^3 \int_0^T \int_{\mathcal{O}} \theta e^{2s\alpha}\varphi^2(\nabla\eta \times (\nabla \times y)) \cdot y dx d\tau, \end{aligned}$$

we obtain with (39) and $\epsilon > 0$:

$$\begin{aligned} s^2\lambda^2 \int_0^T \int_\omega e^{2s\alpha}\varphi|\nabla \times y|^2 dx d\tau \leq {} & \frac{\epsilon}{c_0(\lambda)} \int_Q e^{2s\alpha}|\nabla(\nabla \times y)|^2 dx d\tau \\ & + \frac{\epsilon s^2\lambda^2}{c_0(\lambda)} \int_Q e^{2s\alpha}\varphi^2|\nabla \times y|^2 dx d\tau + C\epsilon^{-1}c_0(\lambda)s^4\lambda^4 \int_0^T \int_{\mathcal{O}} e^{2s\alpha}\varphi^3|y|^2 dx d\tau. \end{aligned}$$

Then with (51) for $\epsilon > 0$ small, for $\lambda \geq \lambda_0$, $s \geq c_1 \max(s_0(\lambda), c_0(\lambda)^2\lambda^{-8}e^{2\lambda(1-\frac{1}{k})\|\eta\|_\infty})$ and $c_1 > 0$, $\lambda_0 > 0$, $C > 0$ large enough we obtain:

$$\begin{aligned} s^{-1} \int_Q e^{2s\alpha}\varphi^{-1}|\Delta y|^2 dx d\tau + \int_Q e^{2s\alpha}|\nabla(\nabla \times y)|^2 dx d\tau + s\lambda^2 \int_Q e^{2s\alpha}\varphi|\nabla y|^2 dx d\tau \\ + s^2\lambda^2 \int_Q e^{2s\alpha}\varphi^2|\nabla \times y|^2 dx d\tau + s^3\lambda^4 \int_Q e^{2s\alpha}\varphi^3|y|^2 dx d\tau \leq \\ c_0(\lambda)C\Big(I_1(s,\lambda,f) + c_0(\lambda)s^4\lambda^4 \int_0^T \int_{\mathcal{O}} e^{2s\alpha}\varphi^3|y|^2 dx d\tau\Big). \end{aligned} \tag{52}$$

Step 7.
First, an obvious calculation gives

$$\begin{aligned} s^{-1}\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1}|\partial_t y+\frac{1}{\varepsilon}\nabla(\nabla\cdot y)|^2dxd\tau \\ = s^{-1}\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1}(|\partial_t y|^2+\frac{1}{\varepsilon^2}|\nabla(\nabla\cdot y)|^2)dxd\tau \\ +2(\varepsilon s)^{-1}\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1}(\partial_t y\cdot\nabla(\nabla\cdot y))dxd\tau, \end{aligned}$$

and with

$$\begin{aligned} 2(s\varepsilon)^{-1}\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1}(\partial_t y\cdot\nabla(\nabla\cdot y))dxd\tau = \\ -2(s\varepsilon)^{-1}\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1}(\partial_t(\nabla\cdot y)(\nabla\cdot y))dxd\tau \\ = -(s\varepsilon)^{-1}\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1}\partial_t(|\nabla\cdot y|^2)dxd\tau \\ = 2\varepsilon^{-1}\int_Q(\widehat{\alpha}_t)e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1}|\nabla\cdot y|^2dxd\tau \\ +(s\varepsilon)^{-1}\int_Q e^{2s\widehat{\alpha}}(\widehat{\varphi})_t\widehat{\varphi}^{-2}|\nabla\cdot y|^2dxd\tau \\ \le C\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}|\varepsilon^{-\frac{1}{2}}\nabla\cdot y|^2dxd\tau, \end{aligned}$$

where in the last step we have used (42) and (40), we obtain:

$$\begin{aligned} s^{-1}\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1}|\partial_t y+\frac{1}{\varepsilon}\nabla(\nabla\cdot y)|^2dxd\tau \ge \\ s^{-1}\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1}(|\partial_t y|^2+\frac{1}{\varepsilon^2}|\nabla(\nabla\cdot y)|^2)dxd\tau - C\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}|\varepsilon^{-\frac{1}{2}}\nabla\cdot y|^2dxd\tau. \end{aligned}$$

Moreover, for $\epsilon>0$ we have

$$\begin{aligned} s\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}|\varepsilon^{-\frac{1}{2}}\nabla\cdot y|^2dxd\tau = s\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}\nabla(\varepsilon^{-1}\nabla\cdot y)\cdot ydxd\tau \\ \le \epsilon s^{-1}\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1}|\frac{1}{\varepsilon^2}\nabla(\nabla\cdot y)|^2dxdt+\frac{s^3}{\epsilon}\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^3|y|^2dxd\tau \end{aligned}$$

and by combining the two above inequalities for $\epsilon>0$ small enough we obtain for $s>s_0$ and $s_0>0$, $C_0>0$ large enough:

$$\begin{aligned} s^{-1}\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1}(|\partial_t y|^2+\frac{1}{\varepsilon^2}|\nabla(\nabla\cdot y)|^2)dxd\tau+s\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}|\varepsilon^{-\frac{1}{2}}\nabla\cdot y|^2dxd\tau \quad \le \\ C_0 s^{-1}\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1}|\partial_t y+\frac{1}{\varepsilon}\nabla(\nabla\cdot y)|^2dxd\tau+s^3\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^3|y|^2dxd\tau. \end{aligned}$$

Then since $\partial_t y + \frac{1}{\varepsilon}\nabla(\nabla\cdot y) = -\Delta y - f_0 - f_1 - \sum_{j=1}^n \left((\nabla f_{1,j})z_{1,j} + {}^t(\nabla f_{2,j})z_{2,j}\right)$, with (39) we obtain for s large enough:

$$s^{-1}\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1}|\partial_t y + \frac{1}{\varepsilon}\nabla(\nabla\cdot y)|^2 dx d\tau \leq C\left(s^{-1}\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1}|\Delta y|^2 dx d\tau + I_1(s,\lambda,f)\right)$$

and since $e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1} \leq e^{2s\alpha}\varphi^{-1}$ for s large, we conclude by combining the two above inequalities with (52).

6. **Proof of Theorem 2.3.** The proof is inspired from a now standard duality argument, see [9, Lemma. 2.1] about the heat equation. In what follows we suppose that $\lambda > \lambda_0$, $s \geq \widehat{s}_0(\lambda)$ for λ_0 and $s_0(\lambda)$ given in Theorem 2.2. First, we introduce the following fourth order problem:

$$\left\{\begin{array}{rcll} \mathcal{L}(s\varphi e^{2s\alpha}\mathcal{L}^* w) + s^3\lambda^4 e^{2s\alpha}\varphi^3 w \mathbf{1}_{\mathcal{O}} + s^4\lambda^4 e^{2s\alpha}\varphi^3 y & = & 0 & \text{in } Q,\\ w = 0, \quad s\varphi e^{2s\alpha}\mathcal{L}^* w & = & 0 & \text{on } \Sigma,\\ \varphi e^{2s\alpha}\mathcal{L}^* w(0) = \varphi e^{2s\alpha}\mathcal{L}^* w(0) & = & 0 & \text{in } \Omega, \end{array}\right. \tag{53}$$

where we have used the notations $\mathcal{L}w = \partial_t w - \Delta w - \frac{1}{\varepsilon}\nabla(\nabla\cdot w)$ and $\mathcal{L}^* w = -\partial_t w - \Delta w - \frac{1}{\varepsilon}\nabla(\nabla\cdot w)$. Problem (53) admits a unique solution w such that $I_0(s,\lambda,w) + s^{-1}\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}^{-1}\frac{1}{\varepsilon^2}|\nabla(\nabla\cdot w)|^2 dx d\tau < +\infty$. Indeed, if we define the following symmetric bilinear form on $P_0 \stackrel{\text{def}}{=} \{w \in (C^2(Q))^d \mid\ w = 0 \text{ on } \Sigma\}$,

$$b(w,w') = s\int_Q \varphi e^{2s\alpha}\mathcal{L}^* w\cdot\mathcal{L}^* w dx d\tau + s^4\lambda^4\int_0^T\int_{\mathcal{O}} e^{2s\alpha}\varphi^3 w\cdot w' dx d\tau \quad \forall (w,w') \in P_0^2,$$

as well as the following linear form on P_0,

$$l(w') = -s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3 y\cdot w' dx d\tau \quad \forall w' \in P_0,$$

and if we denote by P the completion of P_0 for the norm induced by $b(\cdot,\cdot)$, then (16) guarantees the coercivity and the continuity of $b(\cdot,\cdot)$ on P, as well as the continuity of l on P. Then by invoking Lax-Milgram Lemma we obtain a unique solution $w \in P$ to

$$b(w,w') = l(w'), \quad \forall w' \in P,$$

satisfying $\|w\|_P \leq \widehat{c}_0(\lambda)^2\|l\|_{P'}$, which means that w satisfies (53) and

$$s\int_Q e^{2s\alpha}\varphi|\mathcal{L}^* w|^2 dx d\tau + s^4\lambda^4\int_0^T\int_{\mathcal{O}} e^{2s\alpha}\varphi^3|w|^2 dx d\tau \leq \widehat{c}_0(\lambda)^2 s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|y|^2 dx d\tau. \tag{54}$$

Then if we set $z=-s\varphi e^{2s\alpha}\mathcal{L}^*w$ and $u=s^4\lambda^4 e^{2s\alpha}\varphi^3 w\mathbf{1}_{\mathcal{O}}$ we obtain that

$$\left\{\begin{array}{rcll} \partial_t z-\Delta z-\dfrac{1}{\varepsilon}\nabla(\nabla\cdot z) &=& s^3\lambda^4 e^{2s\alpha}\varphi^3 y+u\mathbf{1}_{\mathcal{O}} & \text{in } Q,\\ z &=& 0 & \text{on } \Sigma,\\ z(0)=z(T) &=& 0 & \text{on } \Omega, \end{array}\right.$$

and (54) becomes:

$$s^{-1}\int_Q e^{-2s\alpha}\varphi^{-1}|z|^2dxd\tau+s^{-4}\lambda^{-4}\int_0^T\int_{\mathcal{O}} e^{-2s\alpha}\varphi^{-3}|u|^2dxd\tau\leq \widehat{c}_0(\lambda)^2 s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|y|^2dxd\tau. \tag{55}$$

Moreover, if we multiply the first equation in (11) where $f=F_0+\nabla\cdot F_1$ by z and integrate by parts one gets

$$s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|y|^2dxd\tau=\int_Q F_0\cdot zdxd\tau-\int_Q F_1:\nabla zdxd\tau-\int_0^T\int_{\mathcal{O}} u\cdot ydxd\tau. \tag{56}$$

Let us prove (18) first. By multiplying by $s^{-3}\lambda^{-2}e^{-2s\alpha}\varphi^{-3}z$ the equation satisfied by z and integrating by parts we obtain

$$\begin{aligned} &s^{-3}\lambda^{-2}\int_Q e^{-2s\alpha}\varphi^{-3}|\nabla z|^2dxd\tau+s^{-3}\lambda^{-2}\int_Q e^{-2s\alpha}\varphi^{-3}|\varepsilon^{-\frac12}\nabla\cdot z|^2dxd\tau=\\ &+s^{-3}\lambda^{-2}\int_0^T\int_{\mathcal{O}} e^{-2s\alpha}\varphi^{-3}u\cdot zdxd\tau-s^{-3}\lambda^{-2}\int_Q(z\,{}^t\nabla(e^{-2s\alpha}\varphi^{-3})):\nabla zdxd\tau\\ &-\varepsilon^{-\frac12}s^{-3}\lambda^{-2}\int_Q(\nabla(e^{-2s\alpha}\varphi^{-3})\cdot z)(\varepsilon^{-\frac12}\nabla\cdot z)dxd\tau\\ &\frac12 s^{-3}\lambda^{-2}\int_Q(e^{-2s\alpha}\varphi^{-3})_t|z|^2dxd\tau+s\lambda^2\int_Q y\cdot zdxd\tau \end{aligned}$$

and with $s^{-3}\lambda^{-2}|(e^{-2s\alpha}\varphi^{-3})_t|\leq Cs^{-2}\lambda^{-2}e^{-2s\alpha}\varphi^{-1}$ and $s^{-3}\lambda^{-2}|\nabla(e^{-2s\alpha}\varphi^{-3})|\leq Cs^{-2}\lambda^{-1}e^{-2s\alpha}\varphi^{-2}$ we deduce that for $\lambda>\lambda_0$, $s\geq C\widehat{s}_0(\lambda)$ and $C>0$ large enough:

$$\begin{aligned} &s^{-3}\lambda^{-2}\int_Q e^{-2s\alpha}\varphi^{-3}|\nabla z|^2dxd\tau+s^{-3}\lambda^{-2}\int_Q e^{-2s\alpha}\varphi^{-3}|\varepsilon^{-\frac12}\nabla\cdot z|^2dxd\tau\leq\\ &C\varepsilon^{-1}s^{-1}\int_Q e^{-2s\alpha}\varphi^{-1}|z|^2dxd\tau+s^3\lambda^4\int_Q e^{-2s\alpha}\varphi^3|y|^2dxd\tau\\ &+s^{-4}\lambda^{-4}\int_0^T\int_{\mathcal{O}} e^{-2s\alpha}\varphi^{-3}|u|^2dxd\tau \end{aligned}$$

and with (55) one has:

$$\varepsilon s^{-3}\lambda^{-2}\int_Q e^{-2s\alpha}\varphi^{-3}|\nabla z|^2dxd\tau\leq C\widehat{c}_0(\lambda)^2 s^3\lambda^4\int_Q e^{-2s\alpha}\varphi^3|y|^2dxd\tau.$$

Then the above inequality with (56) and (55) yields:

$$s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|y|^2dxd\tau\leq C\widehat{c}_0(\lambda)^2\Big(s\int_Q e^{2s\alpha}\varphi|F_0|^2dxd\tau+\frac{s^3\lambda^2}{\varepsilon}\int_Q e^{2s\alpha}\varphi^3|F_1|^2dxd\tau+s^4\lambda^4\int_0^T\int_{\mathcal{O}} e^{2s\alpha}\varphi^3|y|^3dxd\tau\Big). \tag{57}$$

Thus, we multiply by $s\lambda^2 e^{2s\alpha}\varphi y$ the equation satisfied by y ((11) with (17)) and integrate by parts we obtain

$$\begin{aligned}
&s\lambda^2\int_Q e^{2s\alpha}\varphi|\nabla y|^2 dxd\tau + s\lambda^2\int_Q e^{2s\alpha}\varphi|\varepsilon^{-\frac{1}{2}}\nabla\cdot y|^2 dxd\tau = \\
&+ s\lambda^2\int_Q e^{2s\alpha}\varphi F_0\cdot y dxd\tau - s\lambda^2\int_Q e^{2s\alpha}\varphi F_1 : \nabla y dxd\tau \\
&- \frac{1}{2}s\lambda^2\int_Q (e^{2s\alpha}\varphi)_t|y|^2 dxd\tau - s\lambda^2\int_Q (y\,{}^t\nabla(e^{2s\alpha}\varphi)) : \nabla y dxd\tau \\
&- \varepsilon^{-\frac{1}{2}}s\lambda^2\int_Q (\nabla(e^{2s\alpha}\varphi)\cdot y)(\varepsilon^{-\frac{1}{2}}\nabla\cdot y)dxd\tau
\end{aligned}$$

with $s\lambda^2|(e^{2s\alpha}\varphi)_t| \leq Cs^2\lambda^2 e^{2s\alpha}\varphi^3$ and $s\lambda^2|\nabla(e^{2s\alpha}\varphi)| \leq Cs^2\lambda^3 e^{2s\alpha}\varphi^2$ we deduce that for $\lambda > \lambda_0$, $s \geq C\widehat{s}_0(\lambda)$ and $C > 0$ large enough:

$$\begin{aligned}
&s\lambda^2\int_Q e^{2s\alpha}\varphi|\nabla y|^2 dxd\tau + s\lambda^2\int_Q e^{2s\alpha}\varphi|\varepsilon^{-\frac{1}{2}}\nabla\cdot y|^2 dxd\tau \leq \\
&C\Big(\frac{s^3\lambda^4}{\varepsilon}\int_Q e^{2s\alpha}\varphi^3|y|^2 dxd\tau + s^{-1}\int_Q e^{2s\alpha}\varphi^{-1}|F_0|^2 dxd\tau + s\lambda^2\int_Q e^{2s\alpha}\varphi|F_1|^2 dxd\tau\Big).
\end{aligned}$$

Finally, combining the above inequality with (57) yields (18).

Let us now prove (19). By multiplying by $s^{-\frac{5}{2}}\lambda^{-2}e^{-2s\alpha^*}\varphi^{*-3}z$ the equation satisfied by z and integrating by parts (by taking into account that $e^{-2s\alpha^*}\varphi^{*-3}$ does not depend on the space variable) we first obtain

$$\begin{aligned}
s^{-\frac{5}{2}}\lambda^{-2}\int_Q e^{-2s\alpha^*}\varphi^{*-3}|\nabla z|^2 dxd\tau \quad &+ \quad s^{-\frac{5}{2}}\lambda^{-2}\int_Q e^{-2s\alpha^*}\varphi^{*-3}|\varepsilon^{-\frac{1}{2}}\nabla\cdot z|^2 dxd\tau \\
= \frac{1}{2}s^{-\frac{5}{2}}\lambda^{-2}\int_Q (e^{-2s\alpha^*}\varphi^{*-3})_t|z|^2 dxd\tau \quad &+ \quad s^{\frac{1}{2}}\lambda^2\int_Q e^{2s(\alpha-\alpha^*)}\varphi^3\varphi^{*-3}y\cdot z dxd\tau \\
&+ \quad s^{-\frac{5}{2}}\lambda^{-2}\int_0^T\int_{\mathcal{O}} e^{-2s\alpha^*}\varphi^{*-3}u\cdot z dxd\tau
\end{aligned}$$

and with $e^{-2s\alpha^*} \leq e^{-2s\alpha}$, $\varphi^{*-3} \leq \varphi^{-3}$ and $|(e^{-2s\alpha^*}\varphi^{*-3})_t| \leq Cs\varphi^{*-1}e^{-2s\alpha^*} \leq C\varphi^{-1}se^{-2s\alpha}$ and (55) one gets for $\lambda > \lambda_0$, $s \geq C\widehat{s}_0(\lambda)$ and $C > 0$ large enough:

$$s^{-\frac{5}{2}}\lambda^{-2}\int_Q e^{-2s\alpha^*}\varphi^{*-3}|\nabla z|^2 dxd\tau \leq C\widehat{c}_0(\lambda)^2 s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|y|^2 dxd\tau.$$

Then the above inequality with (55) and (56) yields:

$$\begin{aligned}
s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|y|^2 dxd\tau \leq C\widehat{c}_0(\lambda)^2\Big(&s^4\lambda^4\int_0^T\int_{\mathcal{O}} e^{2s\alpha}\varphi^3|y|^3 dxd\tau \\
&s\int_Q e^{2s\alpha}\varphi|F_0|^2 dxd\tau + s^{\frac{5}{2}}\lambda^2\int_Q e^{2s\alpha^*}\varphi^{*3}|F_1|^2 dxd\tau\Big).
\end{aligned} \tag{58}$$

Moreover, multiplying by $s^2\lambda^2 e^{2s\widehat{\alpha}}\widehat{\varphi}y$ the equation satisfied by y and integrating in space by parts yields,

$$s^2\lambda^2\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}|\nabla y|^2 dxd\tau + s^2\lambda^2\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}|\varepsilon^{-\frac{1}{2}}\nabla\cdot y|^2 dxd\tau = \\ -\frac{s^2\lambda^2}{2}\int_Q (e^{2s\widehat{\alpha}}\widehat{\varphi})_t|y|^2 dxd\tau + s^2\lambda^2\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}F_0\cdot y dxd\tau \\ - s^2\lambda^2\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}F_1 : \nabla y dxd\tau,$$

and with $e^{2s\widehat{\alpha}} \le e^{2s\alpha}$, $\widehat{\varphi} \le \varphi$ and $|(e^{2s\widehat{\alpha}}\widehat{\varphi})_t| \le Cse^{2s\alpha}\varphi^3$, for $\lambda > \lambda_0$, $s \ge C\widehat{s}_0(\lambda)$ and $C > 0$ large enough we obtain:

$$s^2\lambda^2\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}|\nabla y|^2 dxd\tau + s^2\lambda^2\int_Q e^{2s\widehat{\alpha}}\widehat{\varphi}|\varepsilon^{-\frac{1}{2}}\nabla\cdot y|^2 dxd\tau \le \\ C\left(s^3\lambda^4\int_Q e^{2s\alpha}\varphi^3|y|^2 dxd\tau + s\int_Q e^{2s\alpha}\varphi|F_0|^2 dxd\tau + s^2\lambda^2\int_Q e^{2s\alpha}\varphi|F_1|^2 dxd\tau\right).$$

Finally, combining the above inequalities with (58) yields (19).

7. **Proof of Theorem 3.1, 3.2 and 3.3.** First, analogously as what is done in [8] about the Oseen equation, if we combine (16) with uniform a priori estimates for the nonhomogeneous penalized Oseen system (see [1]), we obtain that the solution of

$$\left\{\begin{array}{rcll} -\partial_t y - \Delta y - (\nabla y)z - {}^t(\nabla y)z - \frac{1}{2}(\nabla\cdot z)y & & & \\ +\frac{1}{2}\nabla(z\cdot y) - \frac{1}{\varepsilon}\nabla(\nabla\cdot y) & = & g & \text{in } Q, \\ y & = & 0 & \text{on } \Sigma, \\ y(T) & = & y_T & \text{in } \Omega, \end{array}\right. \tag{59}$$

satisfies the following inequality:

$$\int_Q e^{2s\beta}\gamma|\nabla y|^2 dxd\tau + \int_Q e^{2s\beta}\gamma^3|y|^2 dxd\tau + \|y(0)\|^2_{(L^2(\Omega))^d} \le \\ C_0\left(\int_Q e^{2s\beta}\varphi|g|^2 dxd\tau + \int_0^T\int_{\mathcal{O}} e^{2s\beta}\gamma^3|y|^3 dxd\tau\right), \tag{60}$$

for a constant C_0 which is independent on ε. Thus, for $(f, v_0) \in F_0 \times (L^2(\Omega))^d$ we denote by (v_{f,v_0}, h_{f,v_0}) the pair minimizing

$$\frac{1}{2}\int_Q e^{-2s\beta}\gamma^{-1}|v|^2 dxd\tau + \frac{1}{2}\int_0^T\int_{\mathcal{O}} e^{-2s\beta}\gamma^{-3}|h|^2 dxd\tau,$$

over all (v, h) solution to

$$\left\{\begin{array}{rcll} \partial_t v - \Delta v + B(z,v) + B(v,z) - \frac{1}{\varepsilon}\nabla(\nabla\cdot v) & = & f + h\mathbf{1}_{\mathcal{O}} & \text{in } Q, \\ v & = & 0 & \text{on } \Sigma, \\ v(0) = v_0, \qquad v(T) & = & 0 & \text{in } \Omega. \end{array}\right. \tag{61}$$

By (60) we have that the above problem is well-posed and that:

$$\int_Q e^{-2s\beta}\gamma^{-1}|v_{f,v_0}|^2dxd\tau+\int_0^T\int_{\mathcal{O}}e^{-2s\beta}\gamma^{-3}|h_{f,v_0}|^2dxd\tau\leq C\left(\|e^{-2s\widehat{\beta}}\widehat{\gamma}^{-1}f\|^2_{L^2(0,T;(H^{-1}(\Omega))^d)}+\|v_0\|^2_{(L^2(\Omega))^d}\right). \tag{62}$$

Indeed, if we define the following symmetric bilinear form on $P_0\overset{\text{def}}{=}\{w\in(C^2(Q))^d\mid w=0\text{ on }\Sigma\}$,

$$b_z(w,w')=\int_Q\gamma e^{2s\beta}\mathcal{L}_z^*w\cdot\mathcal{L}_z^*w'dxd\tau+\int_0^T\int_{\mathcal{O}}e^{2s\beta}\gamma^3w\cdot w'dxd\tau\quad\forall(w,w')\in P_0^2,$$

where $\mathcal{L}_z^*$ denotes the linear partial differential operator at the left of the first equality in (59), if we define the following linear form on P_0,

$$l_{f,v_0}(w')=\int_0^T\langle f|w'\rangle_{(H_0^1(\Omega)^d)',(H_0^1(\Omega))^d}dt+\int_\Omega v_0\cdot w'(0)dx\quad\forall w'\in P_0,$$

and if we denote by P the completion of P_0 for the norm induced by $b_z(\cdot,\cdot)$, then (60) guarantees the coercivity and the continuity of $b_z(\cdot,\cdot)$ on P, and with $f\in F_0$ and $v_0\in(L^2(\Omega))^d$ one has that l_{f,v_0} is continuous on P. Then by invoking Lax-Milgram Lemma we obtain a unique solution $w\in P$ to

$$b_z(w,w')=l_{f,v_0}(w'),\quad\forall w'\in P,$$

satisfying $\|w\|_P\leq C\|l_{f,v_0}\|_{P'}$, which exactely means that $v_{f,v_0}=\gamma e^{2s\beta}\mathcal{L}_z^*w$ and $h_{f,v_0}=-e^{-2s\beta}\gamma^{-3}w$ satisfy (61) (in a transposition sense) and that (62) holds. Moreover, we verify that $e^{-s\frac{\widehat{\beta}}{2}}\widehat{\gamma}^{-\frac14}v_{f,v_0}$ satisfies (61) for a nonhomogeneous right term

$$e^{-s\frac{\widehat{\beta}}{2}}\widehat{\gamma}^{-\frac14}f+(e^{-s\frac{\widehat{\beta}}{2}}\widehat{\gamma}^{-\frac14})_tv_{f,v_0}+e^{-s\frac{\widehat{\beta}}{2}}\widehat{\gamma}^{-\frac14}h_{f,v_0}\mathbf{1}_{\mathcal{O}}\in L^2(0,T;(H^{r-1}(\Omega))^d)$$

and combining uniform a priori estimates for (61) and (62) one obtains for all $r\in[0,1]$:

$$\begin{aligned}\|e^{-s\frac{\widehat{\beta}}{2}}\widehat{\gamma}^{-\frac14}v_{f,v_0}\|_{L^2(0,T;(H^{r+1}(\Omega))^d)}+\|e^{-s\frac{\widehat{\beta}}{2}}\widehat{\gamma}^{-\frac14}v_{f,v_0}\|_{C([0,T];(H^r(\Omega))^d)}\\+\|e^{-s\beta}\gamma^{-\frac12}v_{f,v_0}\|_{(L^2(Q))^d}+\|e^{-s\beta}\gamma^{-\frac32}h_{f,v_0}\|_{(L^2((0,T)\times\mathcal{O}))^d}\leq\\C(\|e^{-s\widehat{\beta}}\widehat{\gamma}^{-\frac12}f\|^2_{L^2(0,T;(H^{r-1}(\Omega))^d)}+\|v_0\|_{r,\varepsilon}).\end{aligned}$$

Then Theorem 3.1 is proved.

Next, from classical estimates of Navier-Stokes type nonlinearity one has for $r\in[\frac{d-2}{2},1]$, $r\neq0$,

$$\|B(w_1,w_2)\|_{(H^{r-1}(\Omega))^d}\leq C\Big(\|w_1\|_{(H^r(\Omega))^d}\|w_2\|_{(H^{r+1}(\Omega))^d}+\|w_2\|_{(H^r(\Omega))^d}\|w_1\|_{(H^{r+1}(\Omega))^d}\Big),$$

which yields

$$\begin{aligned}\|B(u,u)\|_{F_r}&\leq C\|u\|^2_{W_r},\\\|B(u_1,u_1)-B(u_2,u_2)\|_{F_r}&\leq C(\|u_1\|_{W_r}+\|u_2\|_{W_r})\|u_1-u_2\|_{W_r}.\end{aligned}$$

If we denote $f(u) \stackrel{\text{def}}{=} -B(u,u)$, for $v_0 \in (H_0^r(\Omega))^d$ the mapping $\Psi : (u,h) \mapsto (v_{f(u),v_0}, h_{f(u),v_0})$ is continuous from E_r into itself and verifies that:

$$\begin{aligned} \|\Psi(u,h)\|_{E_r} &\leq C_1(\|(u,h)\|_{E_r}^2 + \|v_0\|_{r,\varepsilon}), \\ \|\Psi(u_1,h_1) - \Psi(u_2,h_2)\|_{E_r} &\leq C_1(\|u_2\|_{W_r} + \|u_2\|_{W_r})\|(u_2,h_1) - (u_2,h_2)\|_{E_r}. \end{aligned}$$

Then if we denote $B_\delta = \{(u,h) \in E_r \mid \|(u,h)\|_{E_r} \leq \rho\delta\}$ one verifies that for $\|v_0\|_{r,\varepsilon} \leq \delta$ for any $\rho > 0$ and $\mu > 0$ obeying $\rho\mu < \frac{1}{2C_1}$ and $\rho \geq 2\max(C_1, 1)$ the mapping Ψ is a contraction of B_δ into itself and (31) admits a unique solution in B_δ. Then Theorem 3.2 is proved.

Finally, if we denote by $(v_\varepsilon, h_\varepsilon) \in B_\delta$ the solution of (31) obtained above, then for $v_0 = z_0 - \overline{z}_0 \in V_n^0(\Omega)$ we have $\|v_0\|_{r,\varepsilon} = \|v_0\|_{(H^r(\Omega))^d}$ and v_ε is uniformly (in ε) bounded in $L^2(0,T;(H^{1+r}(\Omega))^d)$ and in $C([0,T];(H^r(\Omega))^d)$. Then from a compactness argument we deduce that v_ε is strongly convergent in $(L^2(Q))^d$. Moreover, since v_ε is also weakly convergent in $L^2(0,T;(H^1(\Omega))^d)$ then we deduce that $B(v_\varepsilon, v_\varepsilon) \to (v \cdot \nabla)v$ in $(L^1(Q))^d$. Thus, by combining (31) with the boundedness of $(v_\varepsilon, h_\varepsilon)$ in E_r we deduce that $(\varepsilon^{-1}\nabla \cdot v_\varepsilon)$ is bounded in $L^2(0,T;H^r(\Omega)/\mathbb{R})$. Then $(v_\varepsilon, h_\varepsilon, \varepsilon^{-1}\nabla \cdot v_\varepsilon)$ converges weakly to (v,h,p) in $E_r \times H^r(\Omega)/\mathbb{R}$ and by passing to the limit in (31) we obtain that $(\overline{z}, h, \overline{\pi}) = (v,h,p) + (z,0,\pi)$ satisfies (36). Then Theorem 3.3 is proved.

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