LOCAL STABILIZATION OF THE NAVIER–STOKES EQUATIONS WITH A FEEDBACK CONTROLLER LOCALIZED IN AN OPEN SUBSET OF THE DOMAIN

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□ *We study the local exponential stabilization, near a given steady-state flow, of solutions of the 2-D and 3-D Navier–Stokes equations in a bounded domain. The control is effectuated through a distributed control localized in an open subset of the domain. We apply a linear feedback controller, obtained by the solution of a control problem. Lyapunov functionals are given for a large class of cost functionals.*

Keywords Feedback stabilization; Lyapunov functional; Navier–Stokes system; Riccati equation.

AMS Subject Classification 76D05; 76D55; 93D15; 35B40; 35Q30.

1. INTRODUCTION

Let Ω be a bounded and connected domain in $\mathbb{R}^d$ for $d = 2$ or $d = 3$, with a boundary $\Gamma = \partial\Omega$ of class C^3, and composed of N connected components $\Gamma^{(1)}, \ldots, \Gamma^{(N)}$. Let us consider a stationary motion of an incompressible fluid in Ω described by a velocity field z_s and a pressure p_s. The pair (z_s, p_s) is a solution to the stationary Navier–Stokes equations:

$$-\nu\Delta z_s + (z_s \cdot \nabla)z_s + \nabla r_s = f, \quad \nabla \cdot z_s = 0 \text{ in } \Omega \quad \text{and} \quad z_s = v_b \text{ on } \Gamma. \quad (1.1)$$

In this setting, $\nu > 0$ is the viscosity, f is a function in $\mathbf{L}^2(\Omega)$, v_b belongs to $\mathbf{H}^{\frac{3}{2}}(\Gamma)$ and obeys $\int_{\Gamma^{(j)}} v_b \cdot n = 0$, for all $j = 1, \ldots, N$, where n denotes the unit normal vector to Γ, exterior to Ω. Notice that here and in the following, we write in bold the spaces of vector fields: $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$, $\mathbf{H}^{\frac{3}{2}}(\Gamma) = (H^{\frac{3}{2}}(\Gamma))^d$, etc. We recall that a solution to (1.1) is known to exist in $\mathbf{H}^2(\Omega) \times H^1(\Omega)/\mathbb{R}$ (see [12, Chap. VIII, Thm. 4.1 and Thm. 5.2]).

If z_s is an unstable equilibrium state, and if we assume that at time $t = 0$ the velocity is equal to $z_0 \neq z_s$, then even if z_0 is close to z_s, the resulting unsteady velocity $z(t)$ when $t > 0$ will not necessarily stay close to z_s. Hence, in order that $z(t)$ go back to z_s as $t \to \infty$, we are interested in finding a feedback control that is localized in an open subset $\omega \subset \Omega$, and that is of the form $u(t) = I_\omega F(z_s - z(t))$ for $t \geq 0$. Here, $I_\omega : \mathbf{L}^2(\Omega) \longrightarrow \mathbf{L}^2(\Omega)$ is defined by $(I_\omega v)(x) = m(x)v(x)$ for all $x \in \Omega$ where m is the characteristic function of the open subset ω. With such a feedback controller, the unsteady velocity and pressure (z, r) satisfy the instationary closed loop Navier–Stokes equations:

$$\partial_t z - \nu\Delta z + (z \cdot \nabla)z + \nabla r = I_\omega F(z_s - z) + f \quad \text{and}$$
$$\nabla \cdot z = 0 \quad \text{in } (0, \infty) \times \Omega, \tag{1.2}$$
$$z = v_b \text{ on } (0, \infty) \times \Gamma, \quad z(0) = z_0. \tag{1.3}$$

Obtaining an adequate law F can be reduced to studying an auxiliary optimal control problem. The idea is the following. We linearize (1.2)–(1.3) around (z_s, r_s) and we replace the feedback term in (1.2) with the unknown $I_\omega u \in L^2(0, \infty; \mathbf{L}^2(\Omega))$, where u belongs to $L^2(0, \infty; \mathbf{L}^2(\Omega))$. Then we obtain the linearized system

$$\partial_t y - \nu\Delta y + (y \cdot \nabla)z_s + (z_s \cdot \nabla)y + \nabla p = I_\omega u \quad \text{in } (0, \infty) \times \Omega, \tag{1.4}$$
$$\nabla \cdot y = 0 \text{ in } (0, \infty) \times \Omega, \quad y = 0 \text{ on } (0, \infty) \times \Gamma, \quad y(0) = y_0. \tag{1.5}$$

Under some detectability condition, the solution of the minimizing problem

$$\inf\left\{ \frac{1}{2}\int_0^\infty \|\mathscr{C}y(t)\|_{\mathcal{Z}}^2 dt + \frac{1}{2}\int_0^\infty \int_\Omega |u(t)|^2 dt \,\middle|\, (y, u) \text{ satisfies (1.4)–(1.5)} \right\} \tag{1.6}$$

provides a feedback law $F = \Pi$, which is defined in an adequate space of initial conditions, and the linear operator Π satisfies an algebraic Riccati equation. Notice that the observation operator $\mathscr{C} : \mathscr{D}(\mathscr{C}) \subset \mathbf{L}^2(\Omega) \to \mathcal{Z}$ is not necessarily bounded in $\mathbf{L}^2(\Omega)$. Moreover, the controller $u(t) = -I_\omega \Pi y(t)$ for all $t \geq 0$, is known to stabilize the linear system (1.4)–(1.5) around zero ([13, Chap. 2]). Hence, when $F = \Pi$, the main difficulty is to prove that the global stabilization of the linear system (1.4)–(1.5) induces the local stabilization of the nonlinear system (1.2)–(1.3) around

z_s. Let us define $y = z - z_s$, where z_s and z are the respective solution to systems (1.1) and (1.2)–(1.3). Then y is solution to

$$\partial_t y - \nu\Delta y + (y\cdot\nabla)z_s + (z_s\cdot\nabla)y + (y\cdot\nabla)y + \nabla p = -I_\omega \Pi y \quad \text{in } (0,\infty)\times\Omega, \tag{1.7}$$

$$\nabla\cdot y = 0 \ \text{ in } (0,\infty)\times\Omega, \quad y = 0 \ \text{ on } (0,\infty)\times\Gamma, \quad y(0) = y_0, \tag{1.8}$$

and the nonlinear stabilization problem we address is the following one: if $y_0 = z_0 - z_s$ belongs to a neighborhood of zero in a space of initial conditions, how can we prove that y goes to zero when t goes to infinity?

Because the Riccati operator Π is positive and definite, a first strategy consists in searching sufficient conditions on the observation operator $\mathscr{C}$ in (1.6), so that the value function of the problem defined by $\xi \mapsto \langle \Pi\xi|\xi\rangle$, is a Lyapunov function for the system (1.7)–(1.8). This path is followed in [3, 6] for a distributed control when $d = 3$ and in [5] for a Dirichlet boundary control. In [3, 6] the observation operator $\mathscr{C}$ is chosen to be equal to $A^{\frac{3}{4}}$, where A is the Stokes operator. A Riccati operator Π sufficiently "unbounded" is obtained, and it allows one to estimate the nonlinear term in the Navier–Stokes system. Then one has

$$\langle \Pi\xi \mid \xi\rangle \sim \|\xi\|^2_{\mathbf{H}_0^{1/2}(\Omega)} \quad \forall \xi \in V_0^{1/2}(\Omega) = \{y \in \mathbf{H}_0^{1/2}(\Omega) \mid \nabla\cdot y = 0\},$$

and by composing (1.7) with Πy, it is proved that, if the initial condition y_0 is sufficiently small in $V_0^{1/2}(\Omega)$, then the mapping $t \mapsto \langle \Pi y(t) \mid y(t)\rangle$ exponentially decreases to 0 as t goes to infinity. Another strategy relies in a fixed point method that uses the regularity properties of the coupled system obtained from the optimality condition of (1.6). By this way, the cost functional can be chosen as simple as possible. One may choose $\mathscr{C}$ equal to the identity in $\mathbf{L}^2(\Omega)$ for instance. This path is followed in [4, 16] to stabilize the two-dimensional Navier–Stokes system with a feedback Dirichlet boundary control, but no Lyapunov function is given there. Then a natural question is how can we obtain a Lyapunov function when no special constraints are imposed in the cost functional by the observation operator?

In the current paper, by generalizing an idea already used in [1, 2], we propose a method to obtain a Lyapunov function for the system (1.7)–(1.8), for a large family of observation operators. In particular, we are going to show that the degree of unboundedness of the observation operator $\mathscr{C}$ does not affect the final stabilization result. Assume $d = 3$ and $\alpha \in [0, \frac{3}{4}]$, and consider the Riccati operator Π that is given by the optimal control problem

$$\inf\left\{\frac{1}{2}\int_0^\infty \|A^\alpha y(t)\|^2_{\mathbf{L}^2(\Omega)}dt + \frac{1}{2}\int_0^\infty \int_\Omega |u(t)|^2 dt \,\middle|\, (y,u) \text{ satisfies (1.4)–(1.5)}\right\}. \tag{1.9}$$

Thus, by introducing the closed loop linear operator A_Π, which is associated with system (1.7)–(1.8), and the operator $\Pi^{(s)} = A_\Pi^{*\frac{s}{2}+\frac{1}{2}-\alpha}\Pi A_\Pi^{\frac{s}{2}+\frac{1}{2}-\alpha}$, we show that if y_0 is small in $V_0^s(\Omega)$ (defined below), then the mapping $\xi \mapsto \langle \Pi^{(s)}\xi \mid \xi\rangle$ is a Lyapunov function for the system (1.7)–(1.8). With such an approach, it seems to be useless to choose an observation operator that is unbounded. It is sufficient to restrict our choice to $\alpha = 0$. However, we make α varying in $[0, \frac{3}{4}]$ in order to fully understand the role of the observation operator in (1.9). Notice that the case $\alpha = \frac{3}{4}$ is treated in [3].

The paper is organized as follows. In Section 2, we give an abstract formulation of (1.4)–(1.5) and we state our main local stabilization result. Section 3 is dedicated to the study of the optimal control problem that provides a feedback controller $F = \Pi$ solution to an algebraic Riccati equation. In Section 4, we apply this feedback law to the nonlinear system and we give a proof of a local stabilization result.

2. PRELIMINARIES AND MAIN RESULTS

2.1. Notations

Let X and Y be two Banach spaces. If A is a closed linear mapping in X, we denote its domain by $\mathcal{D}(A)$. Moreover, we denote by $\mathcal{L}(X, Y)$ the space of all bounded operators from X to Y, and we use the shorter expression $\mathcal{L}(X) = \mathcal{L}(X, X)$. For $0 < T \le \infty$, the space $L^2(0, T; X)$ is the well-known Lebesgue space and we also define:

$$W(0,T;X,Y) = \left\{ y \in L^2(0,T;X) \,\middle|\, \frac{dy}{dt} \in L^2(0,T;Y)\right\}.$$

When $T \in (0, +\infty)$, and if X is continuously and densely embedded in Y, then the space $W(0, T; X, Y)$ is continuously embedded in $C([0, T]; [X, Y]_{\frac{1}{2}})$ (see [7, Rem. 4.1, p. 95 and Prop. 4.3, p. 99]).

Next, let us recall that Ω is a bounded and connected domain in $\mathbb{R}^d$, for $d = 2$ or $d = 3$, with a boundary $\Gamma = \partial\Omega$ of class C^3, and composed of N connected components $\Gamma^{(1)}, \dots, \Gamma^{(N)}$. We will use the usual function spaces $L^2(\Omega)$, $H^s(\Omega)$, $H_0^s(\Omega)$ and $H^{-s}(\Omega) = (H_0^s(\Omega))'$, and we write in bold the spaces of vector fields $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$, $\mathbf{H}^s(\Omega) = (H^s(\Omega))^d$, $\mathbf{H}_0^s(\Omega) = (H_0^s(\Omega))^d$, and $\mathbf{H}^{-s}(\Omega) = (H^{-s}(\Omega))^d$. The norms are denoted by $\|\cdot\|_{X(\Omega)}$, where the subscript $X(\Omega)$ refers to the space that is considered, and we denote the scalar product in $\mathbf{L}^2(\Omega)$ by $(\cdot|\cdot)$. Moreover, if $y \in \mathbf{L}^2(\Omega)$ is such that $\nabla \cdot y \in L^2(\Omega)$, then we denote the normal trace of y in $H^{-\frac{1}{2}}(\Gamma)$ by $y \cdot n$ (see [11, III. 3]).

Thus, we introduce the spaces of free divergence functions:

$$V_n^s(\Omega) = \{y \in \mathbf{H}^s(\Omega) \mid \nabla \cdot y = 0 \text{ in } \Omega,\ y \cdot n = 0 \text{ on } \Gamma\}, \quad s \in [0, 2].$$

We define the interpolation space:

$$V_0^s(\Omega) = [V_n^2(\Omega) \cap \mathbf{H}_0^1(\Omega), V_n^0(\Omega)]_{1-s/2}, \quad s \in [0, 2]$$

and

$$V_0^s(\Omega) = V_0^2(\Omega) \cap \mathbf{H}^s(\Omega), \quad s > 2,$$
$$V_0^{-s}(\Omega) = \left(V_0^s(\Omega)\right)', \quad s \geq 0.$$

In this setting, $(V_0^s(\Omega))'$ is the dual space of $V_0^s(\Omega)$ with respect to the pivot space $V_n^0(\Omega)$. It is equipped with the duality pairing $\langle \cdot|\cdot \rangle_{V_0^{-s}(\Omega), V_0^s(\Omega)}$. The following equalities are well-known:

$$V_0^s(\Omega) = V_n^s(\Omega), \quad s \in [0, 1/2[,$$
$$V_0^{1/2}(\Omega) = \left\{ y \in V_n^{1/2}(\Omega) \,\middle|\, \int_\Omega \rho(x)^{-1}|y|^2 < +\infty \right\},$$
$$V_0^s(\Omega) = \{y \in V_n^s(\Omega) \mid y = 0 \text{ on } \Gamma\} \quad \text{if } s > 1/2,$$

where $\rho(x)$ is the distance from x to Γ. Notice that, according to the above definition, the subscript 0 only means that we have vanishing Dirichlet boundary condition.

Next, let us define the spaces of pressures with zero mean:

$$L_0^2(\Omega) = \left\{ p \in L^2(\Omega) \,\middle|\, \int_\Omega p = 0 \right\} \quad \text{and} \quad \mathscr{H}^s(\Omega) = H^s(\Omega) \cap L_0^2(\Omega), \quad s \geq 0.$$

We recall that the following *Helmholtz decomposition* holds,

$$\mathbf{L}^2(\Omega) = V_n^0(\Omega) \oplus \nabla \mathscr{H}^1(\Omega),$$

and we introduce the Leray projector P, which is the orthogonal projector from $\mathbf{L}^2(\Omega)$ onto $V_n^0(\Omega)$ (see [11, Chap. III, Thm. 1.1]). The operator $P \in \mathscr{L}(\mathbf{L}^2(\Omega), V_n^0(\Omega))$ can be extended to a bounded linear operator from $\mathbf{H}^{-1}(\Omega)$ onto $V_0^{-1}(\Omega)$ by

$$Py : w \in V_0^1(\Omega) \longmapsto \langle y \mid w \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)}.$$

About the extension of the Leray projector see [5, App. A].

2.2. Oseen System

This subsection is devoted to the abstract reformulation of (1.4)–(1.5). First, we define the Stokes operator in $V_n^0(\Omega)$ by

$$\mathscr{D}(A) = V_0^2(\Omega) \quad \text{and} \quad Ay = -\nu P\Delta y.$$

It is well-known that $(\mathscr{D}(A), A)$ is nonnegative, self-adjoint and definite, and that its fractional powers A^θ are well defined and satisfy $\mathscr{D}(A^\theta) = V_0^{2\theta}(\Omega)$ for all $\theta \in [0, 1]$ (see [10]). Moreover, it is the infinitesimal generator of an analytic semigroup $(e^{-At})_{t>0}$ on $V_n^0(\Omega)$. Next, we introduce the following trilinear form:

$$b(v_1, v_2, v_3) = \int_\Omega (v_1 \cdot \nabla)v_2 \cdot v_3 \quad \forall (v_1, v_2, v_3) \in \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega),$$

which is known to satisfy the estimate

$$\begin{aligned} b(v_1, v_2, v_3) \le C\|v_1\|_{\mathbf{H}^{s_1}(\Omega)}\|v_2\|_{\mathbf{H}^{1+s_2}(\Omega)}\|v_3\|_{\mathbf{H}^{s_3}(\Omega)} \\ \forall (v_1, v_2, v_3) \in \mathbf{H}^{s_1}(\Omega)\mathbf{H}^{1+s_2}(\Omega)\mathbf{H}^{s_3}(\Omega), \end{aligned} \tag{2.1}$$

where s_1, s_2, and s_3 are real positive numbers such that $s_1 + s_2 + s_3 \ge \frac{d}{2}$ if $s_i \ne \frac{d}{2}$, $i = 1, 2, 3$ or $s_1 + s_2 + s_3 > \frac{d}{2}$ if $s_i = \frac{d}{2}$, for at least one i (see [8, Chap. 6, Prop. 6.1 (6.10)]). Hence, we define the nonlinear mapping

$$\begin{aligned} &N : V_0^1(\Omega) \to V_0^{-1}(\Omega), \\ &\quad \langle N(y) \mid w\rangle_{V_0^{-1}(\Omega), V_0^1(\Omega)} = b(y, y, w) \quad \forall (y, w) \in V_0^1(\Omega) \times V_0^1(\Omega), \end{aligned} \tag{2.2}$$

and the linear operator

$$\begin{aligned} &\mathscr{D}(A_0) = V_0^1(\Omega), \\ &(A_0 y \mid w) = b(y, z_s, w) + b(z_s, y, w) \quad \forall (y, w) \in V_0^1(\Omega) \times V_n^0(\Omega). \end{aligned}$$

The definition of A_0 is consistent because with (2.1) we have

$$|(A_0 y, w)| \le \|y\|_{V_0^1(\Omega)}\|z_s\|_{\mathbf{H}^2(\Omega)}\|w\|_{V_n^0(\Omega)}.$$

Next, for $T > 0$, $s \in [0, 1]$, $f \in L^2(0, T; \mathbf{L}^2(\Omega))$ and $y_0 \in V_n^0(\Omega)$, we consider the following nonlinear system:

$$y' + Ay + A_0 y + N(y) = Pf, \quad y(0) = y_0. \tag{2.3}$$

In the following proposition, we state the partial differential equations corresponding with the system (2.3).

Proposition 2.1. *Let* $(z_s, r_s) \in \mathbf{H}^2(\Omega) \times \mathscr{H}^1(\Omega)$ *be the solution to* (1.1), *let* $T \in (0, \infty)$, $s \in [0, 1]$, $f \in L^2(0, T; \mathbf{L}^2(\Omega))$ *and let* $y_0 \in V_0^s(\Omega)$. *Then the following results hold.*

(i) *If there exits* $y \in W(0, T; V_0^{s+1}(\Omega), V_0^{s-1}(\Omega))$ *solution to the evolution equation*

$$y' + Ay + A_0 y + N(y) = Pf, \quad y(0) = y_0 \in V_0^s(\Omega), \tag{2.4}$$

then there exists a unique $p \in H^{-\frac{1}{2}+\frac{s}{2}}(0, T; \mathscr{H}^s(\Omega))$ *such that* (y, p) *is solution to the system*

$$\partial_t y - \nu\Delta y + (y \cdot \nabla)z_s + (z_s \cdot \nabla)y + (y \cdot \nabla)y + \nabla p = f \quad \text{in } (0, T) \times \Omega, \tag{2.5}$$

$$\nabla \cdot y = 0 \ \text{ in } (0, T) \times \Omega, \ \ y = 0 \ \text{ on } (0, T) \times \Gamma, \quad y(0) = y_0 \in V_0^s(\Omega). \tag{2.6}$$

In this setting, (2.5) *is understood as an equality in the distribution space* $\mathscr{D}'(0, T; \mathbf{H}^{-1}(\Omega))$ *and in* (2.6), *the divergence and the trace conditions are understood as equalities in* $L^2((0, T) \times \Omega)$ *and in* $L^2(0, T; \mathbf{L}^2(\Gamma))$ *respectively.*

(ii) *Conversely, if* $(y, p) \in W(0, T; V_0^{s+1}(\Omega), V_0^{s-1}(\Omega)) \times H^{-\frac{1}{2}+\frac{s}{2}}(0, T; \mathscr{H}^s(\Omega))$ *is a solution to* (2.5)–(2.6), *then* y *satisfies* (2.4).

Proof. (i) By defining $\mathscr{Y}(t) = \int_0^t y(\tau)d\tau$, $\mathscr{N}(\mathscr{Y})(t) = \int_0^t N(y)(\tau)d\tau$ and $\mathscr{F}(t) = \int_0^t f(\tau)d\tau$ and by integrating (2.4) over $(0, t)$ we obtain

$$0 = y - y_0 + (A + A_0)\mathscr{Y} + \mathscr{N}(\mathscr{Y}) - P\mathscr{F} \in H^1(0, T; V_0^{s-1}(\Omega)).$$

Moreover, with $H^1(0, T; V_0^{s-1}(\Omega)) \subset C([0, T]; V_0^{s-1}(\Omega))$, this last equation is verified pointwise in time. Then due to [19, Rem. 1.4(i), Chap. 1, p. 15], at each time $t > 0$, there is a unique $\mathscr{P}(t) \in L_0^2(\Omega)$ that satisfies

$$\begin{aligned}\nabla\mathscr{P}(t) = y(t) - y_0 - \nu\Delta\mathscr{Y}(t) + (\mathscr{Y}(t) \cdot \nabla)z_s + (z_s \cdot \nabla)\mathscr{Y}(t) \\ + \int_0^t (y(\tau) \cdot \nabla)y(\tau)\, d\tau - \mathscr{F}(t) \in \mathbf{H}^{s-1}(\Omega).\end{aligned} \tag{2.7}$$

Hence, by checking each term in the right-hand side of the previous equality, and in particular by remarking that $y \in W(0, T; V_0^{s+1}(\Omega), V_0^{s-1}(\Omega)) \hookrightarrow H^{\frac{1}{2}+\frac{s}{2}}(0, T; \mathbf{L}^2(\Omega))$, we deduce that $\nabla\mathscr{P} \in H^{\frac{1}{2}+\frac{s}{2}}(0, T; \mathbf{H}^{s-1}(\Omega))$. As a consequence, $p = -\frac{d}{dt}\mathscr{P}$ belongs to $H^{-\frac{1}{2}+\frac{s}{2}}(0, T; L_0^2(\Omega))$ and we easily verify that (y, p) is solution to (2.5)–(2.6).

(ii) By applying $P \in \mathscr{L}(\mathbf{H}^{-1}(\Omega), V_0^{-1}(\Omega))$ on (2.5), we obtain the first equation in (2.4) and the conclusion is straightforward. □

Remark 2.2. The technique used in the proof of Proposition 2.1, to obtain a pressure term in $H^{-\frac{1}{2}+\frac{s}{2}}(0,T;\mathscr{H}^s(\Omega))$, is inspired from [19, Chap. III, Prop. 1.1, p. 266 and p. 307]. The pressure p only belongs to such a time negative Sobolev space because we do not have $\partial_t y \in L^2(0,T;\mathbf{H}^{s-1}(\Omega))$ but only $\partial_t y \in L^2(0,T;V_0^{s-1}(\Omega))$. This is deeply due to the fact that Dirichlet boundary conditions have to be considered because Ω is bounded (see [14, Chap. 3, Rem. 3.1, 4)]). Indeed, if $\Omega = \mathbb{R}^d$ then we introduce the spaces

$$V^s(\mathbb{R}^d) = \{y \in \mathbf{H}^s(\mathbb{R}^d) \mid \nabla \cdot y = 0 \text{ in } \mathbb{R}^d\} \quad \text{and} \quad V^{-s}(\mathbb{R}^d) = (V^s(\mathbb{R}^d))' \ \forall s \geq 0,$$

and the orthogonal Leray projector P is defined from $\mathbf{L}^2(\mathbb{R}^d)$ onto $V^0(\mathbb{R}^d)$. Thus, one easily verifies that P commutes with the space derivative operators, and for all $s \in [0,1]$, we have:

$$P(\mathbf{H}^{1-s}(\mathbb{R}^d)) = V^{1-s}(\mathbb{R}^d) \quad \text{and} \quad \|Pv\|_{V^{1-s}(\mathbb{R}^d)} \leq \|v\|_{\mathbf{H}^{1-s}(\mathbb{R}^d)} \ \forall v \in \mathbf{H}^{1-s}(\mathbb{R}^d).$$

Then for the solution y to (2.4), for all $t \geq 0$ we can make the calculation

$$\begin{aligned}\|y(t)\|_{\mathbf{H}^{s-1}(\mathbb{R}^d)} &= \sup_{v \in \mathbf{H}^{1-s}(\mathbb{R}^d)} \frac{(y(t)|v)}{\|v\|_{\mathbf{H}^{1-s}(\mathbb{R}^d)}} = \sup_{v \in \mathbf{H}^{1-s}(\mathbb{R}^d)} \frac{(y(t)|Pv)}{\|v\|_{\mathbf{H}^{1-s}(\mathbb{R}^d)}} \\ &\leq \sup_{v \in \mathbf{H}^{1-s}(\mathbb{R}^d)} \frac{(y(t)|Pv)}{\|Pv\|_{\mathbf{H}^{1-s}(\mathbb{R}^d)}} = \sup_{v \in V^{1-s}(\mathbb{R}^d)} \frac{(y(t)|v)}{\|v\|_{V^{1-s}(\mathbb{R}^d)}} = \|y(t)\|_{V^{s-1}(\mathbb{R}^d)},\end{aligned}$$

and we deduce that $y \in H^1(0,T;\mathbf{H}^{s-1}(\mathbb{R}^d))$ from $y \in H^1(0,T;V^{s-1}(\mathbb{R}^d))$. Thus, (2.7) finally yields $\nabla\mathscr{P} \in H^1(0,T;\mathbf{H}^{s-1}(\mathbb{R}^d))$ and $p \in L^2(0,T;\mathscr{H}^s(\mathbb{R}^d))$. Notice that when Ω is a bounded open subset of $\mathbb{R}^d$, the equality $P(\mathbf{H}_0^{1-s}(\Omega)) = V_0^{1-s}(\Omega)$ is true when $s \in]\frac{1}{2},1]$ and the same technique allows one to improve the regularity of the pressure. In Proposition 2.1, when $s \in]\frac{1}{2},1]$, one can also prove that $p \in L^2(0,T;\mathscr{H}^s(\Omega))$. However, for Ω bounded and $s \in [0,\frac{1}{2}]$, we cannot obtain a result better than $\|y(t)\|_{\mathbf{H}^{s-1}(\Omega)} \geq \|y(t)\|_{V_0^{s-1}(\Omega)}$, and the previous technique does not apply.

Next, in view of defining a linear quadratic minimizing problem, we now consider the linear system

$$y' + Ay + A_0 y = PI_\omega u, \quad y(0) = y_0. \tag{2.8}$$

The unbounded operator $A + A_0$ with domain $\mathscr{D}(A + A_0) = \mathscr{D}(A) = V_0^2(\Omega)$ is the so-called Oseen operator. Because it can be viewed as a perturbation of A with a perturbation term A_0 with domain $\mathscr{D}(A_0) = \mathscr{D}(A^{\frac{1}{2}})$, we obtain the analyticity of $(e^{-(A+A_0)t})_{t>0}$ on $V^0_n(\Omega)$ from the analyticity of $(e^{-At})_{t>0}$ on $V^0_n(\Omega)$ (see [15, Chap. 3, Cor. 2.4]). Moreover, there exists λ_0 such that

$$\langle(\lambda_0 + A + A_0)y \mid y\rangle_{V_0^{-1}(\Omega),V_0^1(\Omega)} \geq \frac{\nu}{2}\|y\|^2_{V_0^1(\Omega)} \quad \forall y \in V_0^1(\Omega),$$

and we have

$$\mathscr{D}((\lambda_0 + A + A_0)^\theta) = \mathscr{D}((\lambda_0 + A + A_0^*)^\theta) = V_0^{2\theta}(\Omega) \quad \forall \theta \in [0, 1], \tag{2.9}$$

(see [17, (4.2), Lem. 4.1]). In this setting, A_0^* is defined by

$$\mathscr{D}(A_0^*) = V_0^1(\Omega), \quad (A_0^* y|w) = b(w, z_s, y) - b(z_s, y, w) \quad \forall (y, w) \in V_0^1(\Omega) \times V_n^0(\Omega).$$

Then, for all $\theta \in [0, 1]$, optimal regularity results ensure that the following mapping

$$\begin{aligned} W(0, T; V_0^{2\theta}(\Omega), V_0^{2(\theta-1)}(\Omega)) &\longrightarrow L^2(0, T; V_0^{2(\theta-1)}(\Omega)) \times V_0^{2\theta-1}(\Omega)) \\ y &\longmapsto (y' + Ay + A_0 y, y(0)) \quad \text{is an isomorphism} \end{aligned} \tag{2.10}$$

(see [7, Chap. 3, Par. 2]). Thus, if we choose a real value $\alpha \in [0, \frac{3}{4}]$, then for $y_0 \in V_0^{2\alpha-1}(\Omega)$ and $u \in L^2(0, T; \mathbf{L}^2(\Omega))$, the linear system (2.8) admits a unique solution $y \in W(0, T; V_0^{2\alpha}(\Omega), V_0^{2(\alpha-1)}(\Omega))$.

In fact, with a boundary of class C^3 and with $z_s \in \mathbf{H}^2(\Omega)$, (2.9) can be generalized in term of norm equivalence:

$$\|(\lambda_0 + A + A_0)^{1+\theta} \cdot \|_{V_n^0(\Omega)} \sim \| \cdot \|_{V_n^{2+2\theta}(\Omega)} \quad \theta \in [-1, 1/2]. \tag{2.11}$$

Indeed, from $\|A_0 y\|_{V_n^{2\theta}(\Omega)} \leq C\|z_s\|_{\mathbf{H}^2(\Omega)}\|y\|_{\mathbf{H}^{2+2\theta}(\Omega)}$ for $\theta \in [0, 1/2]$, we easily deduce that $\|(\lambda_0 + A + A_0)^{1+\theta} y\|_{V_n^0(\Omega)} \leq C\|(\lambda_0 + A + A_0)y\|_{V_n^{2\theta}(\Omega)} \leq C\|y\|_{V_n^{2\theta+2}(\Omega)}$, and conversely, regularity results for the stationary Oseen system, obtained from an easy adaptation of regularity results for the Stoke system (see [11]), yields the continuous embedding $\mathscr{D}((\lambda_0 + A + A_0)^{1+\theta}) \hookrightarrow V_0^{2+2\theta}(\Omega)$. Hence, we obtain the following lemma, which will be useful for the analysis of Section 3.

Lemma 2.3. *For* $\theta \in [-1, 1/2]$ *and* $0 < T < +\infty$, *let* $f \in L^2(0, T; V_0^{2\theta+1}(\Omega))$ *and* $y_0 \in V_0^{2\theta+2}(\Omega)$. *The solution* y *to the equation*

$$y' + \lambda_0 y + Ay + A_0 y = f, \quad y(0) = y_0,$$

belongs to $C(0,T;V_0^{2\theta+2}(\Omega))$*, and obeys*

$$\sup_{t\in[0,T]} \|y(t)\|_{V_0^{2\theta+2}(\Omega))} \le C_\theta\big(\|f\|_{L^2(0,T;V_0^{2\theta+1}(\Omega))} + \|y_0\|_{V_0^{2\theta+2}(\Omega))}\big). \tag{2.12}$$

In this setting, the constant C_θ *does not depend on* T.

Proof. First, for all $0 < T \le +\infty$, it is known that the mapping

$$\Lambda : \begin{cases} W(0,T;V_0^2(\Omega),V_n^0(\Omega)) \to L^2(0,T;V_n^0(\Omega)) \times \mathcal{D}\big((\lambda_0 + A + A_0)^{\frac{1}{2}}\big), \\ \qquad y \mapsto (y' + \lambda_0 y + Ay + A_0 y, y(0)) \end{cases}$$

is an isomorphism (see [7, Chap. 1, Thm. 3.1, (i)]). The operator $-(\lambda_0 + A + A_0)$ is of negative type, and because we have the continuous embedding $W(0,T;V_0^2(\Omega),V_n^0(\Omega)) \hookrightarrow C(0,T;V_0^1(\Omega))$, there exists a constant C independent on T such that

$$\|y\|_{W(0,T;V_0^2(\Omega),V_n^0(\Omega))} \le C(\|f\|_{L^2(0,T;V_n^0(\Omega))} + \|y_0\|_{V_0^1(\Omega))}). \tag{2.13}$$

Finally, when $(f, y_0) \in L^2(0,\infty;V_0^{2\theta+1}(\Omega)) \times V_0^{2\theta+2}(\Omega))$, the estimate (2.12) follows from the equality

$$(\lambda_0 + A + A_0)^{\frac{1}{2}+\theta}\Lambda^{-1}(f,y_0) = \Lambda^{-1}\big((\lambda_0 + A + A_0)^{\frac{1}{2}+\theta} f, (\lambda_0 + A + A_0)^{\frac{1}{2}+\theta} y_0\big),$$

and from the norm equivalence (2.11). □

Remark 2.4. Notice that, in view of studying (1.9), we must have $y \in L^2(0,\infty;V_0^{2\alpha}(\Omega))$. Then with (2.10) the choice of y_0 in $V_0^{2\alpha-1}(\Omega)$ is the natural one. We shall underline that in such a case, the theory of [13, Chap. 1 and Chap. 2] cannot be directly applied. Indeed, with the notations there, one has $Y = V_0^{2\alpha-1}(\Omega)$ equipped with the norm $\|\cdot\|_Y = \|A^{-\frac{1}{2}+\alpha}\cdot\|_{V_n^0(\Omega)}$. Then the observation operator is unbounded in Y and is equal to $R = A^{\frac{1}{2}}$ (remark that $\|Ry\|_Y = \|A^{-\frac{1}{2}+\alpha}Ry\|_{V_n^0(\Omega)} = \|A^\alpha y\|_{V_n^0(\Omega)}$ for $y \in V_0^{2\alpha}(\Omega)$). Hence, the degree of unboundedness of R is equal to $\delta = \frac{1}{2}$, and it does not fit the framework of [13, Chap. 1, Par. 1.8 and Chap. 2, Par. 2.5]. However, for $\varepsilon > 0$ and $y_0 \in V_0^{2\alpha-1+\varepsilon}(\Omega)$, the theory can be applied with $Y = V_0^{2\alpha-1+\varepsilon}(\Omega)$ and $R = A^{\frac{1}{2}-\varepsilon}$.

Remark 2.5. With $y_0 \in V_0^{2\alpha-1}(\Omega)$ and $PI_\omega u \in L^2(0,T;V_n^0(\Omega))$, optimal regularity results for the Oseen system ensures that the set of admissible values for α is $[0,1]$. However, in order to avoid too many technical difficulties, we restrict our study to $\alpha \in [0,\frac{3}{4}]$.

2.3. Main Results

In this paper, we prove the following local stabilization result.

Theorem 2.6. *The following results hold.*

(i) *Let us define the space $\mathscr{X}$ by:*

$$\mathscr{X} = \{L \in \mathscr{L}(V_0^1(\Omega), V_n^0(\Omega)) \mid (L\xi \mid \zeta) = (\xi \mid L\zeta),\ (L\xi \mid \xi) \geq 0,\\ \forall(\xi, \zeta) \in V_0^1(\Omega) \times V_0^1(\Omega)\}. \tag{2.14}$$

Then there is a unique operator $\Pi \in \mathscr{X}$ solution to the algebraic Riccati equation:

$$(\Pi\xi \mid (A + A_0)\zeta) + ((A + A_0)\xi \mid \Pi\zeta) + (I_\omega\Pi\xi \mid I_\omega\Pi\zeta)\\ = (A^\alpha\xi \mid A^\alpha\zeta) \quad \forall(\xi, \zeta) \in V_0^2(\Omega) \times V_0^2(\Omega). \tag{2.15}$$

(ii) *Let $(z_s, r_s) \in \mathbf{H}^2(\Omega) \times \mathscr{H}^1(\Omega)$ be the solution to (1.1) and let Π be the solution to (2.15). For $s \in [\frac{d-2}{2}, 1]$, we consider the system*

$$\partial_t z - \nu\Delta z + (z \cdot \nabla)z + \nabla r = I_\omega\Pi(z_s - z) + f \quad in\ (0, \infty) \times \Omega, \tag{2.16}$$

$$\nabla \cdot y = 0 \ \ in\ (0, \infty) \times \Omega, \quad z = v_b \ \ on\ (0, \infty) \times \Gamma, \quad z(0) = z_0 \in V_0^s(\Omega). \tag{2.17}$$

Then the following result holds. There exist $c > 0$ and $\mu_1 > 0$ such that, if $\delta \in (0, \mu_1)$ and

$$z_0 - z_s \in \mathscr{W}_\delta^s = \{y \in V_0^s(\Omega) \mid \|y\|_{V_0^s(\Omega)} \leq c\delta\}, \tag{2.18}$$

then (2.16)–(2.17) admits a unique solution in the set $\{(z_s, r_s)\} + \mathscr{D}_\delta^s$, where

$$\mathscr{D}_\delta^s = \Big\{(y, p) \in W(0, +\infty; V_0^{s+1}(\Omega), V_0^{s-1}(\Omega)) \times H^{-\frac{1}{2}+\frac{s}{2}}(0, \infty; \mathscr{H}^s(\Omega))\\ \|y\|_{W(0,+\infty;V_0^{s+1}(\Omega),V_0^{s-1}(\Omega))} \leq \delta,\ \|p\|_{H^{-\frac{1}{2}+\frac{s}{2}}(0,+\infty;H^s(\Omega))} \leq \delta(1 + \delta)\Big\}.$$

(iii) *Let us define the unbounded operator $(\mathscr{D}(A_\Pi), A_\Pi) = (V_0^2(\Omega), A + A_0 + PI_\omega\Pi)$, the mapping*

$$V_s : \xi \in V_0^s(\Omega) \longrightarrow \langle\Pi^{(s)}\xi|\xi\rangle_{V_0^{-s}(\Omega), V_0^s(\Omega)} \quad where$$

$$\Pi^{(s)} = A_\Pi^{*\frac{s}{2}+\frac{1}{2}-\alpha}\Pi A_\Pi^{\frac{s}{2}+\frac{1}{2}-\alpha}, \quad s \in \left[\frac{d-2}{2}, 1\right],$$

and for $z_0 \in z_s + \mathscr{W}_\delta^s$, where $\mathscr{W}_\delta^s$ is defined in (2.18), let us consider the solution $(z,p) \in \{(z_s, r_s)\} + \mathscr{D}_\delta^s$ to the system (2.16)–(2.17). Then the function $V_s(\cdot)$ is a Lyapunov function of the system satisfied by $z - z_s$: it obeys

$$V_s(\xi) \sim \|\xi\|^2_{V_0^s(\Omega)} \quad \forall \xi \in V_0^s(\Omega) \text{ and } t \longmapsto V_s(z(t) - z_s) \text{ is decreasing.}$$

Moreover, $t \longrightarrow V_s(z(t)-z_s)$ deacreases exponentially quickly, and there exist $C > 0$ and $\sigma > 0$ such that:

$$\|z(t) - z_s\|_{V_0^s(\Omega)} \leq C\|z_0 - z_s\|_{V_0^s(\Omega)}\, e^{-\sigma t} \quad \forall t \geq 0.$$

Remark 2.7. Equations (2.16) are understood as an equality in the distribution space $\mathscr{D}'(0,\infty;\mathbf{H}^{-1}(\Omega))$ and in (2.17), the divergence and the trace conditions are understood as equalities in $L^2((0,\infty)\times\Omega)$ and in $L^2(0,\infty;\mathbf{L}^2(\Gamma))$, respectively.

Remark 2.8. Notice that a similar result is proved in [3] in the particular case where $\alpha = \frac{3}{4}$.

3. THE LINEAR FEEDBACK LAW

In this section, we exhibit a feedback law Π related to an optimal control problem and which is a solution to an algebraic Riccati equation. It is shown that Π stabilizes the linear Oseen system, and, in view of proving that it also stabilizes the Navier–Stokes system (see Section 4), regularity results for Π or for other related operators are collected.

3.1. Optimal Control Problem Over a Finite Time Horizon

Here, we obtain a linear feedback law Π as the asymptotic value of a mapping $t \mapsto \tilde{\Pi}(t)$. The linear operator $\tilde{\Pi}(t)$ is deduced from an optimal control problem stated over a time horizon $t \in (0,+\infty)$. We also prove some useful regularity results for Π.

First, let us recall that $\alpha \in [0, \frac{3}{4}]$ and let us fix a time horizon $t \in (0,+\infty)$. Hence, given an initial condition $\xi \in V_0^{2\alpha-1}(\Omega)$, we define the control problem

$$\inf\{\mathscr{J}_t(y,u) \mid (y,u) \in W(0,t;V_0^{2\alpha}(\Omega), V_0^{2(\alpha-1)}(\Omega)) \times L^2(0,t;\mathbf{L}^2(\Omega)) \text{ satisfies (3.2)}\} \tag{3.1}$$

where

$$y' + Ay + A_0 y = PI_\omega u, \quad y(0) = \xi \in V_0^{2\alpha-1}(\Omega), \tag{3.2}$$

and where the cost functional $\mathcal{J}_t$ is defined by

$$\mathcal{J}_t(y, u) = \frac{1}{2}\int_0^t \int_\Omega |A^\alpha y|^2 + \frac{1}{2}\int_0^t \int_\Omega |u|^2. \tag{3.3}$$

We recall that a general abstract theory for similar minimizing problems is developed in [13, Chap. 1]. It can be applied if $\xi \in V_0^{2\alpha-1+\varepsilon}(\Omega)$ and $\varepsilon > 0$ (we refer to the explanations given in Remark 2.4). In the following theorem, we state the optimal coupled system that characterizes the solution to (3.1).

Theorem 3.1. *For all $\xi \in V_0^{2\alpha-1}(\Omega)$, the problem (3.1) admits a unique solution $(u_{t,\xi}, y_{t,\xi})$. Moreover, the optimal control obeys $u_{t,\xi} = -I_\omega \Phi_{t,\xi}$ where*

$$(y_{t,\xi}, \Phi_{t,\xi}) \in W(0, t; V_0^{2\alpha}(\Omega), V_0^{2(\alpha-1)}(\Omega)) \times W(0, t; V_0^{2(1-\alpha)}(\Omega), V_0^{-2\alpha}(\Omega))$$

is the unique solution to the following system:

$$(\mathcal{S}_{t,\xi}) \begin{cases} y' + Ay + A_0 y = -PI_\omega \Phi, & y(0) = \xi \in V_0^{2\alpha-1}(\Omega), \\ -\Phi' + A\Phi + A_0^* \Phi = A^{2\alpha} y. & \Phi(t) = 0. \end{cases}$$

Proof. Existence and uniqueness of $(y_{t,\xi}, u_{t,\xi})$ is obvious and we have the optimality condition:

$$D_u \mathcal{J}_t(u_{t,\xi}, y_{t,\xi}) \cdot h = \int_0^t \int_\Omega A^\alpha y_{t,\xi} \cdot A^\alpha w_h + \int_0^t \int_\Omega u_{t,\xi} \cdot h = 0 \quad \forall h \in \mathbf{L}^2(\Omega), \tag{3.4}$$

where $w_h \in W(0, T; V_0^2(\Omega), V_n^0(\Omega))$ satisfies $w_h' + Aw_h + A_0 w_h = PI_\omega h$ and $w_h(0) = 0$. Hence, an integration by part yields the following equivalent expression of (3.4):

$$\int_0^t \int_\Omega I_\omega \Phi_{t,\xi} \cdot h + \int_0^t \int_\Omega u_{t,\xi} \cdot h = 0 \quad \forall h \in \mathbf{L}^2(\Omega), \quad \text{i.e., } u_{t,\xi} = -I_\omega \Phi_{t,\xi},$$

where $\Phi_{t,\xi}$ is the unique solution to the backward system:

$$-\Phi' + A\Phi + A_0 \Phi = A^{2\alpha} y, \quad \Phi(t) = 0.$$

Notice that (2.10) ensures that $\Phi_{t,\xi}$ belongs to $W(0, t; V_0^{2(1-\alpha)}(\Omega), V_0^{-2\alpha}(\Omega))$. □

Remark 3.2. The dependence of $\Phi_{t,\xi}$ with respect to α follows from the fact that $\xi \in V_0^{2\alpha-1}$, but also from the dependence of $A^{2\alpha} y_{t,\xi}$ (see the second line of $(S_{t,\xi})$). Even if $\xi \in V_0^2(\Omega)$, $\Phi_{t,\xi}$ still depends on α by the term $A^{2\alpha} y_{t,\xi}$.

Remark 3.3. Because we have $\Phi_{t,\xi} \in W(0, t; V_0^{2(1-\alpha)}(\Omega), V_0^{-2\alpha}(\Omega))$ from Theorem 3.8, the continuous embedding $W(0, t; V_0^{2(1-\alpha)}(\Omega), V_0^{-2\alpha}(\Omega)) \hookrightarrow C([0, t]; V_0^{1-2\alpha}(\Omega))$ ensures that $\Phi_{t,\xi}(0)$ is well defined in $V_0^{1-2\alpha}(\Omega)$.

For each $t \in (0, +\infty)$, we are now in position to define an operator $\tilde{\Pi}(t)$ associated with (3.1).

Definition 3.4. For all $t \in (0, \infty)$, we define the linear mapping $\tilde{\Pi}(t) \in \mathscr{L}(V_0^{2\alpha-1}(\Omega), V_0^{1-2\alpha}(\Omega))$ by

$$\tilde{\Pi}(t)\xi = \Phi_{t,\xi}(0) \quad \forall \xi \in V_0^{2\alpha-1}(\Omega),$$

where $(y_{t,\xi}, \Phi_{t,\xi}) \in W(0, T; V_0^{2\alpha}(\Omega), V_0^{2(\alpha-1)}(\Omega)) \times W(0, T; V_0^{2(1-\alpha)}(\Omega), V_0^{-2\alpha}(\Omega))$ is the solution to $(S_{t,\xi})$.

Remark 3.5. Because the operator $\tilde{\Pi}$ depends on α, we should use the notation $\tilde{\Pi}_\alpha$. However, for readability convenience, we prefer to write $\tilde{\Pi}$.

Remark 3.6. As a direct consequence of the dynamic programming principle, the operator $\tilde{\Pi}$ also satisfies the following pointwise relationship:

$$\tilde{\Pi}(t-\tau)y_{t,\xi}(\tau) = \Phi_{t,\xi}(\tau) \quad \forall \tau \in (0, t). \tag{3.5}$$

In the following proposition, the existence of an assymptotic value Π of $t \mapsto \tilde{\Pi}(t)$ is stated, as well as useful regularity properties for Π.

Proposition 3.7. *The following results hold.*

(i) *For all $t > 0$, $\tilde{\Pi}(t)$ belongs to $\mathscr{X} \cap \mathscr{L}(V_0^{2\alpha-1}(\Omega), V_0^{1-2\alpha}(\Omega))$ (defined by (2.14)), and for all $(\xi, \zeta) \in V_0^{2\alpha-1}(\Omega) \times V_0^{2\alpha-1}(\Omega)$, the mapping*

$$t \longmapsto \langle \tilde{\Pi}(t)\xi|\xi\rangle_{V_0^{1-2\alpha}(\Omega), V_0^{2\alpha-1}(\Omega)} \tag{3.6}$$

is nondecreasing and obeys:

$$\begin{aligned}\langle \tilde{\Pi}(t)\xi|\xi\rangle_{V_0^{1-2\alpha}(\Omega), V_0^{2\alpha-1}(\Omega)} &= 2\mathcal{J}_t(y_{t,\xi}, u_{t,\xi}) \\ &= 2\inf\{\mathcal{J}_t(y, u) \mid (y, u) \text{ satisfies } (3.2)\}.\end{aligned} \tag{3.7}$$

(ii) *The operator $\tilde{\Pi}$ is uniformly bounded in $\mathscr{L}(V_0^{2\alpha-1}(\Omega), V_0^{1-2\alpha}(\Omega))$:*

$$\tilde{\Pi} \in L^\infty(0, \infty; \mathscr{L}(V_0^{2\alpha-1}(\Omega), V_0^{1-2\alpha}(\Omega))). \tag{3.8}$$

There exists a linear mapping $\Pi \in \mathcal{X} \cap \mathcal{L}(V_0^{2\alpha-1}(\Omega), V_0^{1-2\alpha}(\Omega))$ *such that*

$$\widetilde{\Pi}(t)\xi \longrightarrow \Pi\xi \quad \text{weakly in } V_0^{1-2\alpha}(\Omega) \text{ as } t \to \infty \quad \forall \xi \in V_0^{2\alpha-1}(\Omega). \tag{3.9}$$

(iii) *The pair* $(\widetilde{\Pi}, \Pi)$ *obeys*

$$\widetilde{\Pi} \in L^\infty(0,\infty; \mathcal{L}(V_0^1(\Omega), V_0^{3-4\alpha}(\Omega))) \quad \text{and} \quad \Pi \in \mathcal{L}(V_0^1(\Omega), V_0^{3-4\alpha}(\Omega)), \tag{3.10}$$

and

$$\widetilde{\Pi}(t)\xi \longrightarrow \Pi\xi \quad \text{weakly in } V_0^{3-4\alpha}(\Omega) \text{ as } t \to \infty \ \forall \xi \in V_0^{3-4\alpha}(\Omega). \tag{3.11}$$

(iv) *The following regularizing property holds:*

$$\Pi \in \mathcal{L}(V_0^{2\theta+2\alpha-1}(\Omega), V_0^{2\theta+1-2\alpha}(\Omega)) \quad \forall \theta \in [0, 1-\alpha]. \tag{3.12}$$

Proof. (i) *Optimal cost and boundedness of* $\widetilde{\Pi}(t)$. First, we multiply the first equation in the first line of $(\mathcal{S}_{t,\xi})$ by $\Phi_{t,\zeta}$, we multiply the first equation in the second line of $(\mathcal{S}_{t,\zeta})$ by $y_{t,\xi}$, and we subtract the first resulting equality from the second one. Hence, integrating over $(0,t)$, with $\Phi_{t,\zeta}(t) = 0$, we obtain

$$\langle \widetilde{\Pi}(t)\xi | \zeta \rangle_{V_0^{1-2\alpha}(\Omega), V_0^{2\alpha-1}(\Omega)} = \int_0^t \int_\Omega A^\alpha y_{t,\xi} \cdot A^\alpha y_{t,\zeta} + \int_0^t \int_\Omega I_\omega \Phi_{t,\xi} \cdot I_\omega \Phi_{t,\zeta}, \tag{3.13}$$

and equality (3.7) follows by setting $\xi = \zeta$. Next, the use of the Cauchy–Schwarz inequality in (3.13), the optimality of $(y_{t,\xi}, u_{t,\xi})$ and of $(y_{t,\zeta}, u_{t,\zeta})$, and (2.10) with $\theta = \alpha$ yield:

$$\begin{aligned}
\langle \widetilde{\Pi}(t)\xi | \zeta \rangle_{V_0^{2\alpha-1}(\Omega), V_0^{1-2\alpha}(\Omega)} &\leq 2\mathcal{J}_t(y_{t,\xi}, u_{t,\xi})^{1/2} \mathcal{J}_t(y_{t,\zeta}, u_{t,\zeta})^{1/2} \\
&\leq 2\mathcal{J}_t(e^{-(A+A_0)(\cdot)}\xi, 0)^{1/2} \mathcal{J}_t(e^{-(A+A_0)(\cdot)}\zeta, 0)^{1/2} \\
&\leq C_t \|\xi\|_{V_0^{2\alpha-1}(\Omega)} \|\zeta\|_{V_0^{2\alpha-1}(\Omega)}.
\end{aligned}$$

Then we deduce that $\Pi(t) \in \mathcal{L}(V_0^{2\alpha-1}(\Omega), V_0^{1-2\alpha}(\Omega))$, and $\Pi(t) \in \mathcal{X}$ is a straightforward consequence of (3.13). Finally, for all $t > 0$ and $h > 0$ the optimality of the pair $(y_{t,\xi}, u_{t,\xi})$ yields:

$$\mathcal{J}_t(y_{t,\xi}, u_{t,\xi}) \leq \mathcal{J}_t(y_{t+h,\xi}, u_{t+h,\xi}) \leq \mathcal{J}_{t+h}(y_{t+h,\xi}, u_{t+h,\xi}).$$

It proves that (3.6) is nondecreasing.

(ii) *Asymptotic behavior of* $\widetilde{\Pi}$. According to a null controllability result in [9], there exist $T_0 > 0$ and a pair

$$(\widetilde{y}, \widetilde{u}) \in L^2(0, T_0; \mathscr{D}(A^\alpha)) \times L^2(0, T_0; \mathbf{L}^2(\Omega))$$

solution to (3.2) and which is exactly zero past T_0. Then we have

$$\langle \widetilde{\Pi}(t)\xi|\xi\rangle_{V_0^{1-2\alpha}(\Omega), V_0^{2\alpha-1}(\Omega)} = 2\mathscr{J}_t(u_{t,\xi}, y_{t,\xi}) \le 2\mathscr{J}_{T_0}(\widetilde{u}, \widetilde{y}) < +\infty \quad \forall t > 0,$$

and the existence of $\lim_{t\to+\infty}\langle \widetilde{\Pi}(t)\xi|\xi\rangle_{V_0^{1-2\alpha}(\Omega), V_0^{2\alpha-1}(\Omega)}$ is a consequence of the nondecreasing of (3.6). Next, from

$$\begin{aligned}&\langle \widetilde{\Pi}(t)\xi|\zeta\rangle_{V_0^{1-2\alpha}(\Omega), V_0^{2\alpha-1}(\Omega)}\\ &\quad = \frac{1}{2}\langle \widetilde{\Pi}(t)(\xi+\zeta)|\xi+\zeta\rangle_{V_0^{1-2\alpha}(\Omega), V_0^{2\alpha-1}(\Omega)}\\ &\qquad - \frac{1}{2}\langle \widetilde{\Pi}(t)(\xi-\zeta)|\xi-\zeta\rangle_{V_0^{1-2\alpha}(\Omega), V_0^{2\alpha-1}(\Omega)} \quad \forall(\xi,\zeta) \in V_0^{2\alpha-1}(\Omega) \times V_0^{2\alpha-1}(\Omega)\end{aligned}$$

we deduce that

$$\begin{aligned}&\lim_{t\to+\infty}\langle \widetilde{\Pi}(t)\xi|\zeta\rangle_{V_0^{1-2\alpha}(\Omega), V_0^{2\alpha-1}(\Omega)}\\ &\quad = B(\xi,\zeta) \quad \text{where } B \text{ is bilinear on } V_0^{2\alpha-1}(\Omega) \times V_0^{2\alpha-1}(\Omega).\end{aligned}$$

Moreover, the successive use of the Banach–Steinhaus theorem with the families $(\langle \widetilde{\Pi}(t)\xi|\cdot\rangle_{V_0^{1-2\alpha}(\Omega), V_0^{2\alpha-1}(\Omega)})_{t\ge0}$ and $(\widetilde{\Pi}(t)\xi)_{t\ge0}$ yields (3.8). Finally, by passing to the limit, we conclude that B is continuous on $V_0^{2\alpha-1}(\Omega) \times V_0^{2\alpha-1}(\Omega)$ and that it can be identified with an operator $\Pi \in \mathscr{X} \cap \mathscr{L}(V_0^{2\alpha-1}(\Omega), V_0^{1-2\alpha}(\Omega))$. Then (3.9) is proved.

(iii) *Regularity results* (3.10) *for* $(\widetilde{\Pi}, \Pi)$. To prove (3.10), we use the following expression of $(\widehat{\Phi}_{t,\xi}, \widehat{y}_{t,\xi}) = (e^{-\lambda_0(\cdot)}\Phi_{t,\xi}, e^{-\lambda_0(\cdot)}y_{t,\xi})$:

$$\widehat{y}_{t,\xi}(\tau) = e^{-(A+A_0)\tau}\xi + \int_0^\tau e^{-(A+A_0+\lambda_0)(\tau-s)}e^{-\lambda_0 s}u_{t,\xi}(s)\,ds,$$

$$\widehat{\Phi}_{t,\xi}(\tau) = \int_\tau^t e^{-(A+A_0^*+\lambda_0)(s-\tau)}(2\lambda_0\tilde{\Pi}(\tau-s) + A^{2\alpha})\widehat{y}_{t,\xi}(s)\,ds.$$

If $\alpha \in [0, \frac{1}{2}]$, we first make the following calculation:

$$\|\widetilde{\Pi}(t)\xi\|_{V_0^{2-2\alpha}(\Omega)} = \|\widehat{\Phi}_{t,\xi}(0)\|_{V_0^{2-2\alpha}(\Omega)}$$

$$\begin{aligned}
(\text{by (2.12) with } \theta = -\alpha) \le C_1\big(&\|\widetilde{\Pi}(\tau - \cdot)\widehat{y}_{t,\xi}\|_{L^2(0,t;V_0^{1-2\alpha}(\Omega))} \\
&+ \|A^{\frac{1}{2}+\alpha}\widehat{y}_{t,\xi}\|_{L^2(0,t;V_n^0(\Omega))}\big) \\
(\text{by (3.8) and (2.10)}) \le C_2\big(&\|PI_\omega u_{t,\xi}\|_{L^2(0,t;V_n^0(\Omega))} + \|\xi\|_{V_0^{2\alpha}(\Omega)}\big)
\end{aligned}$$

where $C_1 > 0$ ad $C_2 > 0$ do not depend on t. Then from $\|PI_\omega u_{t,\xi}\|_{L^2(0,t;V_n^0(\Omega))} \le \mathcal{J}_t(y_{t,\xi}, u_{t,\xi})^{1/2}$, from (3.7) and from (3.8), we finally deduce that $\|\widetilde{\Pi}(t)\|_{\mathcal{L}(V_0^{2\alpha}(\Omega), V_0^{2-2\alpha}(\Omega))}$ is bounded independently of $t > 0$. As a consequence, since we have $V_0^{2-2\alpha}(\Omega) \hookrightarrow V_0^{2-4\alpha}(\Omega)$ and $V_0^2(\Omega) \hookrightarrow V_0^{2\alpha}(\Omega)$ for all $\alpha \in [0, \frac{1}{2}]$, we deduce that

$$\sup_{t\ge 0} \|\widetilde{\Pi}(t)\|_{\mathcal{L}(V_0^2(\Omega), V_0^{2-4\alpha}(\Omega))} < +\infty. \tag{3.14}$$

Moreover, when $\alpha \in]\frac{1}{2}, \frac{3}{4}]$, we have $V_0^{1-2\alpha}(\Omega) \hookrightarrow V_0^{2-4\alpha}(\Omega)$, and from (3.8) we deduce that (3.14) also holds when $\alpha \in]\frac{1}{2}, \frac{3}{4}]$. Then for all $\alpha \in [0, \frac{3}{4}]$ we can make the following calculation:

$$\begin{aligned}
\|\widetilde{\Pi}(t)\xi\|_{V_0^{3-4\alpha}(\Omega)} &= \|\widehat{\Phi}_{t,\xi}(0)\|_{V_0^{3-4\alpha}(\Omega)} \\
(\text{by (2.12) with } \theta = 1 - 4\alpha) &\le C_1\big(\|\widetilde{\Pi}(\tau - \cdot)\widehat{y}_{t,\xi}\|_{L^2(0,t;V_0^{2-4\alpha}(\Omega))} \\
&\qquad + \|A\widehat{y}_{t,\xi}\|_{L^2(0,t;V_n^0(\Omega))}\big) \\
(\text{by (3.14)}) &\le C_2\|A\widehat{y}_{t,\xi}\|_{L^2(0,t;V_n^0(\Omega))} \\
(\text{by (2.10) with } \theta = 1) &\le C_3(\|PI_\omega u_{t,\xi}\|_{L^2(0,t;V_n^0(\Omega))} + \|\xi\|_{V_0^1(\Omega)}),
\end{aligned}$$

where $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$ do not depend on t. Then from $\|PI_\omega u_{t,\xi}\|_{L^2(0,t;V_n^0(\Omega))} \le \mathcal{J}_t(y_{t,\xi}, u_{t,\xi})^{1/2}$, from (3.7) and from (3.8), we obtain the first statement in (3.10). Finally, the proof of the second statement in (3.10), and the proof of (3.11) are analogous to the ones of (3.9).

(iv) *Regularizing property* (3.12). From $\Pi \in \mathcal{L}(V_0^{2\alpha-1}(\Omega), V_0^{1-2\alpha}(\Omega))$ and from the second statement in (3.10), we deduce (3.12) by interpolation.

3.2. Optimal Control Problem Over an Infinite Time Horizon

Here, we show that Π is related to an optimal control problem stated over an infinite time horizon. It allows one to prove that the feedback law Π stabilizes the Oseen system, and that it is the solution to an algebraic Riccati equation.

Let us recall that $\alpha \in [0, \frac{3}{4}]$. For an initial condition $\xi \in V_0^{2\alpha-1}(\Omega)$, we consider the following control problem stated over an infinite time horizon:

$$\inf\{\mathcal{J}(y,u)\mid(y,u)\in W(0,\infty;V_0^{2\alpha}(\Omega),V_0^{2(\alpha-1)})\\ \times L^2(0,\infty;\mathbf{L}^2(\Omega))\ \text{ satisfies }\ (3.2)\}, \tag{3.15}$$

where the cost functional $\mathcal{J}$ is defined by

$$\mathcal{J}(y,u)=\frac{1}{2}\int_0^\infty\int_\Omega |A^\alpha y|^2+\frac{1}{2}\int_0^\infty\int_\Omega |u|^2.$$

We recall that a general abstract theory for similar minimizing problems is developed in [13, Chap. 2]. It can be applied if $\xi\in V_0^{2\alpha-1+\varepsilon}(\Omega)$ and $\varepsilon>0$ (we refer to the explanations given in Remark 2.4). The optimal pair (u_ξ,y_ξ) solution to (3.15) is characterized in the following theorem.

Theorem 3.8. *For all $\xi\in V_0^{2\alpha-1}(\Omega)$, the problem* (3.15) *admits a unique solution (y_ξ,u_ξ). The optimal control obeys $u_\xi=-I_\omega\Phi_\xi$ where*

$$(y_\xi,\Phi_\xi)\in W(0,\infty;V_0^{2\alpha}(\Omega),V_0^{2(\alpha-1)}(\Omega))\times C_b([0,\infty[;V_0^{1-2\alpha}(\Omega))$$

is the unique solution to the following system:

$$(\mathcal{S}_\xi)\begin{cases} y'+Ay+A_0y=-PI_\omega\Phi, & y(0)=\xi\in V_0^{2\alpha-1}(\Omega),\\ -\Phi'+A\Phi+A_0^*\Phi=A^{2\alpha}y, & \Phi(\infty)=0,\\ \Phi(t)=\Pi\,y(t)\quad \forall t\geq 0. \end{cases}$$

In this setting, $C_b([0,\infty[;V_0^{1-2\alpha}(\Omega))$ is the space of continuous and bounded functions of $t\in[0,\infty[$ with value in $V_0^{1-2\alpha}(\Omega)$ and the linear operator $\Pi\in\mathcal{L}(V_0^{2\alpha-1}(\Omega),V_0^{1-2\alpha}(\Omega))$ is defined by (3.9). *Moreover, we also have*

$$\langle\Pi\xi|\xi\rangle_{V_0^{1-2\alpha}(\Omega),V_0^{2\alpha-1}(\Omega)}=2\mathcal{J}(y_\xi,u_\xi)=2\inf\{\mathcal{J}(y,u)\mid(y,u)\ \textit{satisfies}\ (3.2)\}. \tag{3.16}$$

Proof. (a) *Existence of solution to* $(\mathcal{S}_\xi)$. According to an exact controllability result stated in [9], the problem (3.15) is solvable and it admits a unique solution (y_ξ,u_ξ). In order to characterize (y_ξ,u_ξ), we first consider the optimal state family $(y_k,\Phi_k)=(y_{k,\xi},\Phi_{k,\xi})\in W(0,k;V_0^{2\alpha}(\Omega),V_0^{2(\alpha-1)}(\Omega))\times W(0,k;V_0^{2(1-\alpha)}(\Omega),V_0^{-2\alpha}(\Omega))$ solution to the system

$$(\mathcal{S}_{k,\xi})\begin{cases} y_k'+Ay_k+A_0y_k=-PI_\omega\Phi_k, & y_k(0)=\xi\in V_0^{2\alpha-1}(\Omega),\\ -\Phi_k'+A\Phi_k+A_0^*\Phi_k=A^{2\alpha}y_k, & \Phi_k(k)=0. \end{cases}$$

Here, the parameter $k\in\mathbb{R}$ is destinated to tend to infinity. In the following, for readability convenience, we drop the subscript ξ and we respectively denote by (u, y) the optimal pair (u_ξ, y_ξ) and by (u_k, y_k) the optimal pair $(u_{k,\xi}, y_{k,\xi})$. Moreover, we extend y_k, Φ_k and u_k by 0 on $(k, +\infty)$ while keeping the same notations y_k, Φ_k and u_k. Then we have $\mathcal{J}(y_k, u_k) = \mathcal{J}_k(y_k, u_k)$, where $\mathcal{J}_k$ is defined in (3.3) with $T = k$, and since (y_k, u_k) minimizes $\mathcal{J}_k$, we deduce that $\mathcal{J}(y_k, u_k) \leq \mathcal{J}(y, u) < +\infty$, and that (y_k, u_k) is bounded in $L^2(0, +\infty; V_0^{2\alpha}(\Omega)) \times L^2(0, +\infty; \mathbf{L}^2(\Omega))$. Then there exists $(\tilde{y}, \tilde{u}) \in L^2(0, +\infty; V_0^{2\alpha}(\Omega)) \times L^2(0, +\infty; \mathbf{L}^2(\Omega))$ such that:

$$(y_k, u_k) \longrightarrow (\tilde{y}, \tilde{u}) \quad \text{weakly in } L^2(0, +\infty; V_0^{2\alpha}(\Omega)) \times L^2(0, +\infty; \mathbf{L}^2(\Omega)).$$

Thus, by passing to the limit in

$$y_k' + Ay_k + A_0 y_k = PI_\omega u_k, \quad y_k(0) = \xi \in V_0^{2\alpha-1}(\Omega), \tag{3.17}$$

we deduce that $\tilde{y}$ is the state associated with the control $\tilde{u}$. Moreover, the lower semicontinuity of $\mathcal{J}$ ensures that $\mathcal{J}(\tilde{y}, \tilde{u}) \leq \liminf_{k\in\mathbb{N}} \mathcal{J}(y_k, u_k) \leq \mathcal{J}(y, u)$, and because (y, u) is optimal, we deduce that $(\tilde{y}, \tilde{u}) = (y, u)$, $\lim_{k\to+\infty} \mathcal{J}_k(y_k, u_k) = \mathcal{J}(y, u)$ and that the following limit holds:

$$(y_k, u_k) \longrightarrow (y, u) \quad \text{strongly in } L^2(0, +\infty; V_0^{2\alpha}(\Omega)) \times L^2(0, +\infty; \mathbf{L}^2(\Omega)). \tag{3.18}$$

Then (3.16) follows from (3.9). Moreover, we deduce that $y \in W(0, +\infty; V_0^{2\alpha}(\Omega), V_0^{2(\alpha-1)}(\Omega))$ from the equality $y' = -Ay - A_0 y - PI_\omega u \in L^2(0, \infty; V_0^{2(\alpha-1)}(\Omega))$. Next, if we recall the equality $\Phi_k(t) = \widetilde{\Pi}(k - t)y_k(t)$ (see (3.5)), then for all $w \in L^2(0, +\infty; V_n^0(\Omega))$ we can write

$$\begin{aligned}\int_0^\infty (\Phi_k(t) - \Pi y(t)|w(t))dt &= \int_0^\infty (\Pi(k-t)(y_k(t) - y(t))|w(t))dt \\ &\quad + \int_0^\infty (((\Pi(k-t) - \Pi)y(t))|w(t))dt.\end{aligned}$$

Thus, from the strong convergence $y_k \to y$ in $L^2(0, +\infty; V_n^{2\alpha}(\Omega))$ and from (3.8) if $\alpha \in [0, \frac{1}{2}]$ or from (3.10) if $\alpha \in [\frac{1}{2}, \frac{3}{4}]$ we deduce that the first integral tend to zero when k goes to infinity. Moreover, from (3.9) if $\alpha \in [0, \frac{1}{2}]$ or from (3.11) if $\alpha \in [\frac{1}{2}, \frac{3}{4}]$ the Lebesgue theorem allows to conclude that the second integral also tend to zero. Then we have the following limit

$$\Phi_k \longrightarrow \Pi y \quad \text{weakly in } L^2(0, +\infty; V_n^0(\Omega)).$$

So we can pass to the limit in (3.17) and we obtain that y satisfies the evolution equation:

$$y' + Ay + A_0 y + PI_\omega \Pi y = 0, \quad y(0) = \xi \in V_0^{2\alpha-1}(\Omega). \tag{3.19}$$

Moreover, from the uniqueness of the solution to (3.19) and from the continuity of $t \to y(t)$ in $V_0^{2\alpha-1}(\Omega)$, we deduce that the mapping

$$T \mapsto S(t)\xi = y(t) \mid y \quad \text{is solution to (3.19)}$$

defines a strongly continuous semigroup on $V_0^{2\alpha-1}(\Omega)$. Then because y belongs to $L^2(0,\infty; V_0^{2\alpha}(\Omega))$, a well-known result due to Datko ensures that the mapping $t \longmapsto \|y(t)\|_{V_0^{2\alpha-1}(\Omega)}$ decreases to zero with an exponential rate (see [15, Chap. 4, Thm. 4.1, p. 116]). Thus, from $\Pi \in \mathscr{L}(V_0^{2\alpha-1}(\Omega), V_0^{1-2\alpha}(\Omega))$ and from $\Phi = \Pi y$, we deduce that

$$\lim_{t\to+\infty} \|\Phi(t)\|_{V_0^{1-2\alpha}(\Omega)} = 0 \quad \text{and} \quad \Phi \in C_b\big([0,\infty[; V_0^{1-2\alpha}(\Omega)\big).$$

Finally, by passing to the limit in $(\mathscr{S}_{k,\xi})$, we can conclude that (y, Φ) is solution to $(\mathscr{S}_\xi)$.

(b) *Uniqueness of solution of* $(\mathscr{S}_\xi)$. Let us suppose that $(y^0, \Phi^0) \in W(0,\infty; V_0^{2\alpha}(\Omega), V_0^{2(\alpha-1)}(\Omega)) \times C_b(0,\infty; V_0^{1-2\alpha}(\Omega))$ satisfies $(\mathscr{S}_0)$. Then the following equality holds:

$$y^0(t) = \int_0^t e^{-(A+A_0+\lambda_0)(t-\tau)}(\lambda_0 y^0(\tau) - PI_\omega \Phi^0(\tau))d\tau \quad \forall t \geq 0.$$

From the Young inequality for convolutions and from the exponential stability of $(e^{-(A+A_0+\lambda_0)t})_{t\geq 0}$ on $V_0^{2\alpha-1}(\Omega)$, we first deduce that

$$\|y^0\|_{L^\infty(0,\infty;V_0^{2\alpha-1}(\Omega))} \leq C(\|y^0\|_{L^2(0,\infty;V_0^{2\alpha-1}(\Omega))} + \|\Phi^0\|_{L^2(0,\infty;V_0^{2\alpha-1}(\Omega))}) < +\infty. \tag{3.20}$$

Thus, we multiply the first line of $(\mathscr{S}_0)$ by Φ^0, we multiply the second line of $(\mathscr{S}_0)$ by y^0, we subtract the second obtained equation to the first one, and by integrating in time over $(0, T)$ for $T \in (0,\infty)$, we finally obtain:

$$\langle \Phi^0(0)|y^0(0)\rangle_{V_0^{2\alpha-1}(\Omega),V_0^{1-2\alpha}(\Omega)} - \langle \Phi^0(T)|y^0(T)\rangle_{V_0^{2\alpha-1}(\Omega),V_0^{1-2\alpha}(\Omega)} = \int_0^T \int_\Omega |A^\alpha y^0|^2 + \int_0^T \int_\Omega |I_\omega \Phi^0|^2. \tag{3.21}$$

Then with (3.20) and $y^0(0) = 0$, and because $\|\Phi^0(T)\|_{V_0^{1-2\alpha}(\Omega)}$ goes to zero when $T \to \infty$, by passing to the limit in (3.21) we obtain the equality:

$$\int_0^\infty \int_\Omega |A^\alpha y^0|^2 + \int_0^\infty \int_\Omega |I_\omega \Phi^0|^2 = 0.$$

As a consequence, we necessary have $y^0 = 0$ and the equality $\Phi^0 = 0$ follows from $\Phi^0 = \Pi y^0$. Then the uniqueness of (y, Φ) is proved. □

Now, let us introduce the linear operator $(\mathcal{D}(A_\Pi), A_\Pi)$ associated with the evolution system (3.19).

Definition 3.9. Let us define the linear operator $(\mathcal{D}(A_\Pi), A_\Pi)$ in $V^0_n(\Omega)$ as follows:

$$\mathcal{D}(A_\Pi) = V^2_0(\Omega) \quad \text{and} \quad A_\Pi y = (A + A_0 + PI_\omega \Pi)y.$$

Some properties of A_Π are collected in the following proposition.

Proposition 3.10. *The following results hold.*

(i) *The unbounded operator $(\mathcal{D}(A_\Pi), A_\Pi)$ is the infinitesimal generator of an analytic and exponentially stable semigroup on $V^0_n(\Omega)$, and for $\xi \in V^{2\alpha-1}_0(\Omega)$, the optimal trajectory y_ξ is the unique solution to:*

$$y' + A_\Pi y = 0, \quad y(0) = \xi. \tag{3.22}$$

(ii) *If $\alpha \in [0, \frac{1}{2}]$, we have the following equalities*

$$\mathcal{D}({A_\Pi}^\theta) = \mathcal{D}({A_\Pi}^{*\theta}) = \mathcal{D}(A^\theta) = V^{2\theta}_0(\Omega) \quad \forall \theta \in [0, 1]. \tag{3.23}$$

(iii) *If $\alpha \in]\frac{1}{2}, \frac{3}{4}]$, we have the following equalities*

$$\mathcal{D}({A_\Pi}^\theta) = V^{2\theta}_0(\Omega) \quad \forall \theta \in [0, 1], \tag{3.24}$$

$$\mathcal{D}({A_\Pi}^{*\theta}) = V^{2\theta}_0(\Omega) \quad \forall \theta \in \left[0, \frac{1}{2}\right[. \tag{3.25}$$

Proof. (i) *Closed loop system.* First, the fact that y is the unique solution to (3.22) is an easy consequence of the existence and of the uniqueness of the solution to $(\mathcal{S}_\xi)$ (where we replace Φ by Πy). Moreover, because y belongs to $L^2(0, \infty; V^0_n(\Omega))$, the exponential stability of $(e^{-A_\Pi t})_{t\geq 0}$ on $V^0_n(\Omega)$ is deduced from Datko's theorem [15, Chap. 4, Thm. 4.1, p. 116]. Finally, if $\alpha \in [0, \frac{1}{2}]$ we have $PI_\omega \Pi \in \mathcal{L}(V^0_n(\Omega))$ and if $\alpha \in]\frac{1}{2}, \frac{3}{4}]$ we have $PI_\omega \Pi \in \mathcal{L}(V^1_0(\Omega), V^0_n(\Omega))$. Then the analyticity of $(e^{-A_\Pi t})_{t\geq 0}$ on $V^0_n(\Omega)$ follows from [15, Chap. 3, Cor. 2.2, p. 81] if $\alpha \in [0, \frac{1}{2}]$, or it follows from [15, Chap. 3, Cor. 2.4, p. 81] if $\alpha \in]\frac{1}{2}, \frac{3}{4}]$.

(ii) *Fractional powers of A_Π when $\alpha \in [0, \frac{1}{2}]$.* First, because we have $PI_\omega \Pi \in \mathcal{L}(V^0_n(\Omega))$, the linear operator A_Π is a bounded perturbation of $A + A_0$ and we have the maximal domains $\mathcal{D}(A_\Pi) = \mathcal{D}(A + A_0) = \mathcal{D}(A)$ and $\mathcal{D}(A^*_\Pi) = \mathcal{D}(A + A^*_0) = \mathcal{D}(A)$. Moreover, from

$$\langle (\lambda_0 + A + A_0)y \mid y \rangle_{V^{-1}_0(\Omega), V^1_0(\Omega)} \geq \frac{\nu}{2} \|y\|^2_{V^1_0(\Omega)},$$

and from

$$(PI_\omega \Pi y|y) \leq \|\Pi\|_{\mathscr{L}(V_n^0(\Omega))} \|y\|^2_{V_n^0(\Omega)} \quad \forall y \in \mathscr{D}(A_\Pi),$$

if we set $\lambda = \lambda_0 + \|\Pi\|_{\mathscr{L}(V_n^0(\Omega))}$, then we easily obtain:

$$((\lambda + A_\Pi)y|y) \geq \frac{\nu}{2}\|y\|^2_{V_0^1(\Omega)} \quad \forall y \in \mathscr{D}(A_\Pi).$$

Then $\lambda + A_\Pi$ is maximal accretive and we deduce the following equalities:

$$\mathscr{D}((\lambda + A_\Pi)^\theta) = [\mathscr{D}(\lambda + A_\Pi), V_n^0(\Omega)]_{1-\theta} = V_0^{2\theta}(\Omega) \quad \forall \theta \in [0,1],$$
$$\mathscr{D}((\lambda + A_\Pi)^{*\theta}) = [\mathscr{D}(\lambda + A_\Pi^*), V_n^0(\Omega)]_{1-\theta} = V_0^{2\theta}(\Omega) \quad \forall \theta \in [0,1],$$

(see [7, Chap. 1, Prop. 6.1]). Then we conclude with $\mathscr{D}((\lambda + A_\Pi)^\theta) = \mathscr{D}(A_\Pi^\theta)$ and $\mathscr{D}((\lambda + A_\Pi)^{*\theta}) = \mathscr{D}(A_\Pi^{*\theta})$ (see [18, Chap. 2, Lem. 2.3.5]).

(iii) *Fractional powers of* A_Π *when* $\alpha \in]\frac{1}{2}, \frac{3}{4}]$. First, from $PI_\omega \Pi \in \mathscr{L}(V_0^1(\Omega), V_n^0(\Omega))$ we deduce that $(A_0 + PI_\omega\Pi)A^{-\frac{1}{2}} \in \mathscr{L}(V_n^0(\Omega))$. Then by choosing $\lambda > 0$ large enough so that

$$\|(A_0 + PI_\omega\Pi)A^{-\frac{1}{2}}\|_{\mathscr{L}(V_n^0(\Omega))}\|A^{\frac{1}{2}}(\lambda + A)^{-1}\|_{\mathscr{L}(V_n^0(\Omega))} < 1/2,$$

we obtain

$$\begin{aligned}\|(\lambda + A)y\|_{V_n^0(\Omega)} &= \|(I + (A_0 + PI_\omega\Pi)(\lambda + A)^{-1})^{-1}(\lambda + A_\Pi)y\|_{V_n^0(\Omega)}\\ &\leq 2\|(\lambda + A_\Pi)y\|_{V_n^0(\Omega)}.\end{aligned}$$

Then the continuous embedding $\mathscr{D}(A_\Pi) \hookrightarrow \mathscr{D}(A)$ holds. Because the converse one is obvious, we deduce that the maximal domain of A_Π is $\mathscr{D}(A_\Pi) = \mathscr{D}(A)$ and that

$$[\mathscr{D}(A_\Pi), V_n^0(\Omega)]_{1-\theta} = [\mathscr{D}(A), V_n^0(\Omega)]_{1-\theta} = V_0^{2\theta}(\Omega). \tag{3.26}$$

Moreover, from the inequalities

$$\langle(\lambda_0 + A + A_0)y \mid y\rangle_{V_0^{-1}(\Omega), V_0^1(\Omega)} \geq \frac{\nu}{2}\|y\|^2_{V_0^1(\Omega)},$$

and

$$(PI_\omega\Pi y \mid y) \leq \|\Pi\|_{\mathscr{L}(V_0^1(\Omega), V_n^0(\Omega))}\|y\|_{V_0^1(\Omega)}\|y\|_{V_n^0(\Omega)} \quad \forall y \in \mathscr{D}(A_\Pi),$$

if we set $\lambda = \lambda_0 + \frac{1}{\nu}\|\Pi\|^2_{\mathscr{L}(V_0^1(\Omega),V_n^0(\Omega))}$, then we easily obtain:

$$((\lambda + A_\Pi)y|y) \geq \frac{\nu}{4}\|y\|^2_{V_0^1(\Omega)} \quad \forall y \in \mathscr{D}(A_\Pi).$$

As a consequence, $\lambda + A_\Pi$ is maximal accretive and the following equalities hold:

$$\mathscr{D}((\lambda + A_\Pi)^\theta) = [\mathscr{D}(\lambda + A_\Pi), V_n^0(\Omega)]_{1-\theta},$$
$$\mathscr{D}((\lambda + A_\Pi)^{*\theta}) = [\mathscr{D}(\lambda + A_\Pi^*), V_n^0(\Omega)]_{1-\theta} \quad \forall \theta \in [0, 1],$$

and

$$\mathscr{D}((\lambda + A_\Pi)^\theta) = \mathscr{D}((\lambda + A_\Pi)^{*\theta}) \quad \forall \theta \in [0, 1/2[,$$

(see [7, Chap. 1, Prop. 6.1]). Then we finally obtain (3.24) and (3.25) from (3.26). □

Remark 3.11. Suppose that $\alpha \in [0, \frac{1}{2}]$. As usual, the extrapolation method permits one to extend the definition of A_Π^{-1} to a bounded linear operator from $(\mathscr{D}(A_\Pi^*))'$ onto $V_n^0(\Omega)$ (see [13, Chap. 0, 0.3]). Then $\|A_\Pi^{-1}\cdot\|_{V_n^0(\Omega)}$ defines a norm that is equivalent to the one of $(\mathscr{D}(A_\Pi^*))' = V_0^{-2}(\Omega)$ (by (3.23) because $\alpha \in [0, \frac{1}{2}]$). More generally, we have the following norm equivalence:

$$\|A_\Pi^{-\theta}\cdot\|_{V_n^0(\Omega)} \sim \|\cdot\|_{V_0^{-2\theta}(\Omega)} \quad \forall \theta \in [0, 1]. \tag{3.27}$$

We now prove that Π can be constructed as the the unique solution to an algebraic Riccati equation.

Theorem 3.12. *Let $\mathscr{X}$ be defined in* (2.14). *Then the operator Π is the unique solution in $\mathscr{X}$ to the following algebraic Riccati equation:*

$$(\Pi\xi|(A + A_0)\zeta) + ((A + A_0)\xi|\Pi\zeta) + (I_\omega\Pi\xi|I_\omega\Pi\zeta)$$
$$= (A^\alpha\xi|A^\alpha\zeta) \quad \forall(\xi, \zeta) \in V_0^2(\Omega) \times V_0^2(\Omega). \tag{3.28}$$

Proof. (i) *Existence.* First, a duality argument allows to deduce that $\Pi \in \mathscr{L}(V_n^0(\Omega), V_0^{-1}(\Omega))$ from $\Pi \in \mathscr{L}(V_0^1(\Omega), V_n^0(\Omega))$. Then from $\mathscr{D}(A_\Pi) = V_0^2(\Omega)$, we obtain

$$y_\xi \in C(0, T; V_0^2(\Omega)) \cap C^1(0, T; V_n^0(\Omega)) \quad \forall \xi \in V_0^2(\Omega), \tag{3.29}$$

$$\Phi_\xi \in C(0, T; V_n^0(\Omega)) \cap C^1(0, T; V_0^{-1}(\Omega)) \quad \forall \xi \in V_0^2(\Omega), \tag{3.30}$$

where $T > 0$ is an arbitrary fixed time horizon. Thus, we multiply the first equality in the first line of $(\mathcal{S}_\xi)$ by Φ_ζ, we multiply the first equality in the second line of $(\mathcal{S}_\xi)$ by y_ζ, and we add the two resulting equations. Moreover, from (3.29) and (3.30), we easily verify that in the resulting equation, the term

$$\begin{aligned}&(y'_\xi(t)|\Phi_\zeta(t)) - \langle\Phi'_\xi(t)|y_\zeta(t)\rangle_{V_0^{-1}(\Omega),V_0^1(\Omega)}\\&\quad = (y'_\xi(t)\,|\,\Pi y_\zeta(t)) - \langle\Pi y'_\xi(t)\,|\,y_\zeta(t)\rangle_{V_0^{-1}(\Omega),V_0^1(\Omega)},\end{aligned}$$

is continuous in time and vanishes. Then we obtain the following equality:

$$\begin{aligned}&(\Phi_\xi(t)|(A+A_0)y_\zeta(t)) + ((A+A_0)y_\xi(t)|\Phi_\zeta(t)) + (I_\omega\Phi_\xi(t)|I_\omega\Phi_\zeta(t))\\&\quad = (A^\alpha y_\xi(t)|A^\alpha y_\zeta(t)) \quad \forall t\in[0,T].\end{aligned}$$

From (3.29) and (3.30), we verify that each term in the previous equality is continuous in time, and we obtain (3.28) by setting $t = 0$.

(ii) *Uniqueness.* The proof is totally analogous to the one of [13, Chap. 2, Thm. 2.4.5]. □

4. STABILIZATION OF THE NAVIER–STOKES SYSTEM

In this section, we prove that for initial conditions belonging to a adequate neighborhood of the origin, the feeback law Π stabilizes the Navier–Stokes system. We recall that Π depends on α, but for readability convenience we prefer to use the notation Π than the notation Π_α.

Let us assume that $s\in[\frac{d-2}{2},1]$ and that $y_0\in V_0^s(\Omega)$. We consider the following nonlinear equation:

$$y' + Ay + A_0y + N(y) + PI_\omega\Pi y = 0, \quad y(0) = y_0, \tag{4.1}$$

where the nonlinear mapping $N : V_0^1(\Omega)\to V_0^{-1}(\Omega)$ is defined by (2.2). In a first step, by using the fact that $-A_\Pi$ is the infinitesimal generator of an analytic semigroup on $V_n^0(\Omega)$ of negative type, we prove that for sufficiently small y_0 in $V_0^s(\Omega)$, there exists a unique solution of (4.1) that belongs to $W(0,\infty; V_0^{1+s}(\Omega), V_0^{-1+s}(\Omega))$.

Theorem 4.1. *Let $s\in[\frac{d-2}{2},1]$. There exist $c_0 > 0$ and $\mu_0 > 0$ such that, if $\delta\in(0,\mu_0)$ and*

$$y_0\in\mathcal{V}_\delta^s = \{y\in V_0^s(\Omega)|\ \|y\|_{V_0^s(\Omega)} < c_0\delta\}, \tag{4.2}$$

system (4.1) admits a unique solution in the set

$$\mathcal{S}^s_\delta = \{y \in W(0,\infty; V^{1+s}_0(\Omega), V^{-1+s}_0(\Omega)) \mid \|y\|_{W(0,\infty;V^{1+s}_0(\Omega),V^{-1+s}_0(\Omega))} \le \delta\}. \tag{4.3}$$

Proof. The proof is divided in two parts. The case $s > 0$ is treated in the first part, and the case $d = 2$ and $s = 0$ is treated in the second part.

(i) *Existence and uniqueness when* $s > 0$. The operator $-A_\Pi$ is the infinitesimal generator of an analytic semigroup on $V^0_n(\Omega)$ of negative type. Then from [7, Chap. 3, Thm. 2.2] where we can set $T = \infty$ (see also [7, Chap. 1, Thm. 3.1(i)]), we deduce that the following mapping is an isomorphism:

$$\begin{aligned} W(0,\infty; V^{1+s}_0(\Omega), V^{-1+s}_0(\Omega)) &\to L^2(0,\infty; V^{-1+s}_0(\Omega)) \times V^s_0(\Omega), \\ y &\mapsto (y' + A_\Pi y,\, y(0)). \end{aligned} \tag{4.4}$$

Hence, we consider the following mapping:

$$\begin{aligned} \Psi : z \in W(0,\infty; V^{1+s}_0(\Omega), V^{-1+s}_0(\Omega)) \to y_z \quad &\text{where } y_z' + A_\Pi y_z = -N(z), \\ &y_z(0) = y_0 \in V^s_0(\Omega), \end{aligned}$$

and we seek $c_0 > 0$ and $\mu_0 > 0$ such that, for every $y_0 \in \mathcal{V}^s_\delta$ with $\delta \in (0,\mu_0)$, Ψ is a contraction in $\mathcal{S}^s_\delta$. Because (4.4) is an isomorphism, from (2.1) with $(s_1, s_2, s_3) = (s, s, 1-s)$, we first obtain the existence of $C_0 > 0$ such that

$$\begin{aligned} &\|\Psi(z)\|_{W(0,\infty;V^{1+s}_0(\Omega),V^{-1+s}_0(\Omega))} \\ &\quad \le C_0(\|z\|_{L^\infty(0,\infty;V^s_0(\Omega))}\|z\|_{L^2(0,\infty;V^{1+s}_0(\Omega))} + \|y_0\|_{V^s_0(\Omega)}. \end{aligned} \tag{4.5}$$

Then the continuous embedding $W(0,\infty; V^{1+s}_0(\Omega), V^{-1+s}_0(\Omega)) \hookrightarrow L^\infty(0,\infty; V^s_0(\Omega))$ provides $C_1 > 0$ such that

$$\|\Psi(z)\|_{W(0,\infty;V^{1+s}_0(\Omega),V^{-1+s}_0(\Omega))} \le C_0(C_1\|z\|^2_{W(0,\infty;V^{1+s}_0(\Omega),V^{-1+s}_0(\Omega))} + \|y_0\|_{V^s_0(\Omega)}).$$

Because we have $z \in \mathcal{S}^s_\delta$ and $y_0 \in \mathcal{V}^s_\delta$, we deduce that

$$\|\Psi(z)\|_{W(0,\infty;V^{1+s}_0(\Omega),V^{-1+s}_0(\Omega))} \le C_0(C_1\mu_0 + c_0)\delta. \tag{4.6}$$

Next, if z_1 and z_2 belong to S^s_δ, then we easily verify that $y = \Psi(z_1) - \Psi(z_2)$ satisfies

$$y' + A_\Pi y = b(z_1 - z_2, z_1) + b(z_2, z_1 - z_2), \quad y(0) = 0.$$

Because (4.4) is an isomorphism, from (2.1) with $(s_1, s_2, s_3) = (s, s, 1-s)$, we obtain $C_2 > 0$ such that

$$\begin{aligned}
&\|\Psi(z_1) - \Psi(z_2)\|_{W(0,\infty;V_0^{1+s}(\Omega),V_0^{-1+s}(\Omega))} \\
&\quad \le C_2\big(\|z_1 - z_2\|_{L^\infty(0,\infty;V_0^s(\Omega))}\|z_1\|_{L^2(0,\infty;V_0^{1+s}(\Omega))} \\
&\qquad + \|z_2\|_{L^\infty(0,\infty;V_0^s(\Omega))}\|z_1 - z_2\|_{L^2(0,\infty;V_0^{1+s}(\Omega))}\big). \qquad (4.7)
\end{aligned}$$

As a consequence, with the continuous embedding $W(0,\infty; V_0^{1+s}(\Omega), V_0^{-1+s}(\Omega) \hookrightarrow L^\infty(0,\infty; V_0^s(\Omega))$ and the fact that z_1 and z_2 both belong to S_δ^s, we obtain the existence of $C_3 > 0$ such that:

$$\|\Psi(z_1) - \Psi(z_2)\|_{W(0,\infty;V_0^{1+s}(\Omega),V_0^{-1+s}(\Omega))} \le C_2 C_3 \mu_0 \|z_1 - z_2\|_{W(0,\infty;V_0^{1+s}(\Omega),V_0^{-1+s}(\Omega))}. \qquad (4.8)$$

Then, if we choose $\mu_0 = \min(\frac{1}{2C_0C_1}, \frac{1}{2C_2C_3})$ and $c_0 < \frac{1}{2C_0}$, from (4.6) and from (4.8), we deduce that Ψ is a contraction in S_δ^s, and that (4.1) admits a unique solution.

(ii) *Existence and uniqueness in the two-dimensional case when* $s = 0$. For all $z \in W(0,\infty; V_0^1(\Omega), V_0^{-1}(\Omega))$, and $v \in V_0^1(\Omega)$, an integration by part yields

$$b(z(t), z(t), v) = -b(z(t), v, z(t)).$$

Thus, from (2.1) with $(s_1, s_2, s_3) = (\frac{1}{2}, 0, \frac{1}{2})$ and from the interpolation inequality

$$\|\cdot\|_{V_0^{1/2}(\Omega)} \le C\|\cdot\|^{1/2}_{V_n^0(\Omega)}\|\cdot\|^{1/2}_{V_0^1(\Omega)},$$

we deduce that $\|N(z)(t)\|_{V_0^{-1}(\Omega)} \le C\|z(t)\|_{V_n^0(\Omega)}\|z(t)\|_{V_0^1(\Omega)}$, which proves (4.5) when $s = 0$. A similar argument also yields (4.7) when $s = 0$. Finally, we can conclude as in the case where $s > 0$. □

Next, it remains to prove that the solution to (4.1) decreases to zero exponentially quickly in $V_0^s(\Omega)$. The proof relies in an adequate choice of scalar product for $V_0^s(\Omega)$, which will provide a Lyapunov function for the system (4.1). We first need to define the following operator.

Definition 4.2. For $s \in [0,1]$, we define the following linear operator:

$$\Pi^{(s)} : V_0^s(\Omega) \longrightarrow V_0^{-s}(\Omega) \quad \text{and} \quad \Pi^{(s)} = A_\Pi^{*\frac{s}{2}+\frac{1}{2}-\alpha}\, \Pi\, A_\Pi^{\frac{s}{2}+\frac{1}{2}-\alpha}.$$

Lemma 4.3. *For $s \in [0,1]$, the operator $\Pi^{(s)}$ obeys*

$$\Pi^{(s)} \in \mathscr{L}(V_0^{2\theta+s}(\Omega), V_0^{2\theta-s}(\Omega)) \quad \begin{cases} \forall \theta \in [0,1/2] & \text{if } s \in]0,1], \\ \forall \theta \in [0,1/2[& \text{if } s = 0. \end{cases} \tag{4.9}$$

Proof. It is an easy consequence of (3.23), (3.24) and of (3.25) with the following calculation:

$$\begin{aligned} \|\Pi^{(s)}\xi\|_{V_0^{2\theta-s}(\Omega)} &= \|A_\Pi^{*\frac{s}{2}+\frac{1}{2}-\alpha}\,\Pi\, A_\Pi^{\frac{s}{2}+\frac{1}{2}-\alpha}\xi\|_{V_0^{2\theta-s}(\Omega)} \\ (\text{because } \theta - s/2 < 1/2) &\le C\|A_\Pi^{*\theta+\frac{1}{2}-\alpha}\Pi A_\Pi^{\frac{s}{2}+\frac{1}{2}-\alpha}\xi\|_{V_n^0(\Omega)} \\ (\text{because } \theta + 1/2 - \alpha < 1/2) &\le C\|\Pi A_\Pi^{\frac{s}{2}+\frac{1}{2}-\alpha}\xi\|_{V_0^{1+2\theta-2\alpha}(\Omega)} \\ (\text{by } (3.12)) &\le C\|A_\Pi^{\frac{s}{2}+\frac{1}{2}-\alpha}\xi\|_{V_0^{2\theta+2\alpha-1}(\Omega)} \\ (\text{by } (3.23) \text{ or } (3.24)) &\le C\|\xi\|_{V_0^{s+2\theta}(\Omega)}. \end{aligned}$$

Thus, from (3.16), we will prove that the bilinear form $(\cdot|\cdot)_{\Pi,s}$ defined by

$$(\xi|\zeta)_{\Pi,s} = \langle \Pi^{(s)}\xi|\zeta\rangle_{V_0^{-s}(\Omega),V_0^s(\Omega)} \quad \forall(\xi,\zeta) \in V_0^s(\Omega) \times V_0^s(\Omega), \tag{4.10}$$

is a scalar product on $V_0^s(\Omega)$ for which A_Π is accretive. In fact, from the algebraic Riccati equation (3.28), it can be shown that

$$(A_\Pi\xi|\xi)_{\Pi,s} \ge \sigma(\xi|\xi)_{\Pi,s} \quad \text{where } \sigma > 0.$$

As a consequence, by using the new scalar product (4.10) with equation (3.22), we deduce that the mapping

$$\xi \longmapsto V_s(\xi) = (\xi|\xi)_{\Pi,s},$$

is a Lyapunov function for the closed loop linear system (3.22), and that $V_s(y(t))$ has an exponential rate of decrease equal to $2\sigma > 0$, for y solution to (3.22). Then analogously, provided that the initial condition y_0 is small enough in $V_0^s(\Omega)$, one can also prove that $V_s(\cdot)$ is a Lyapunov function for the nonlinear system (4.11). The proof relies on an adequate estimate of $(N(y)|y)_{\Pi,s}$, which holds for $s \ge \frac{d-2}{2}$. Hence, let us prove the following two preliminary lemmas.

Lemma 4.4. *For all $s \in [0,1]$, the bilinear form $(\cdot|\cdot)_{\Pi,s}$ defined by (4.10) is a scalar product on $V_0^s(\Omega)$. If we define $\|\xi\|_{\Pi,s} = \big((\xi|\xi)_{\Pi,s}\big)^{1/2}$, then the following norm equivalence holds:*

$$\|\cdot\|_{\Pi,s} \sim \|\cdot\|_{V_0^s(\Omega)}. \tag{4.11}$$

Moreover, we also have:

$$(A_\Pi \cdot|\cdot)_{\Pi,s} \sim \|\cdot\|^2_{V_0^{1+s}(\Omega)}. \tag{4.12}$$

Proof. From (3.23), (3.24) and from the equality $\|\xi\|_{\Pi,s} = \|A_\Pi^{\frac{s}{2}}\xi\|_{\Pi,0}$ for all $\xi \in \mathscr{D}(A_\Pi^{\frac{s}{2}})$, we deduce that proving (4.11) for $s \in [0,1]$ can be reduced to proving (4.11) for $s=0$. First, the existence of $C_1 > 0$ such that $\|\cdot\|_{\Pi,0} \leq C_1\|\cdot\|_{V_n^0(\Omega)}$ is a straightforward consequence of $\Pi^{(0)} \in \mathscr{L}(V_n^0(\Omega))$ (see (4.9) when $\theta = s = 0$). Next, the existence of $C_2 > 0$ such that $\|\cdot\|_{\Pi,0} \geq C_2\|\cdot\|_{V_n^0(\Omega)}$ follows from the following calculation (where $\xi \in V_n^0(\Omega)$ and $\zeta = A_\Pi^{\frac{1}{2}-\alpha}\xi$):

$$\begin{aligned}
\|\xi\|^2_{V_n^0(\Omega)} &= \|A_\Pi^{\alpha-\frac{1}{2}}\zeta\|^2_{V_n^0(\Omega)}\\
\text{(by (3.27) if } \alpha \leq 1/2) &\leq C_1'\|\zeta\|^2_{V_0^{2\alpha-1}(\Omega)}\\
&\leq C_2'\|y_\zeta\|^2_{W(0,+\infty;V_0^{2\alpha}(\Omega),V_0^{2(\alpha-1)}(\Omega))}\\
&= C_2'\big(\|y_\zeta\|^2_{L^2(0,+\infty;V_0^{2\alpha}(\Omega))}\\
&\qquad + \|y_\zeta'\|^2_{L^2(0,+\infty;V_0^{2(\alpha-1)}(\Omega))}\big)\\
\text{(with } y_\zeta' = -Ay_\zeta - A_0y_\zeta - PI_\omega\Pi y_\zeta) &\leq C_3'\big(\|y_\zeta\|^2_{L^2(0,+\infty;V_0^{2\alpha}(\Omega))}\\
&\qquad + \|PI_\omega\Pi y_\zeta\|^2_{L^2(0,+\infty;V_0^{2(\alpha-1)}(\Omega))}\big)\\
&\leq C_4'\big(\|y_\zeta\|^2_{L^2(0,+\infty;\mathscr{D}(A^\alpha))} + \|u_\zeta\|^2_{L^2(0,+\infty;\mathbf{L}^2(\Omega))}\big)\\
&= 2C_4'\mathscr{J}(y_\zeta, u_\zeta)\\
\text{(by (3.16))} &\leq C_2\langle\Pi\zeta|\zeta\rangle_{V_0^{1-2\alpha}(\Omega),V_0^{2\alpha-1}(\Omega)}\\
&= \langle\Pi A_\Pi^{\frac{1}{2}-\alpha}\xi|A_\Pi^{\frac{1}{2}-\alpha}\xi\rangle_{V_0^{1-2\alpha}(\Omega),V_0^{2\alpha-1}(\Omega)}\\
&= \|\xi\|^2_{\Pi,0}.
\end{aligned}$$

Next, to prove (4.12), we extend the validity of (3.28) for all $(\xi,\zeta) \in V_0^{2\alpha}(\Omega) \times V_0^{2\alpha}(\Omega)$ by density, and we replace ξ and ζ by $A_\Pi^{\frac{s}{2}+\frac{1}{2}-\alpha}\xi \in V_0^{2\alpha}(\Omega)$ in (3.28). Then we obtain the following expression of $(A_\Pi\xi|\xi)_{\Pi,s}$:

$$\begin{aligned}
(A_\Pi\xi|\xi)_{\Pi,s} &= \langle A_\Pi\xi|\Pi^s\xi\rangle_{V_0^{s-1}(\Omega),V_0^{1-s}(\Omega)}\\
&= \frac{1}{2}\|A^\alpha A_\Pi^{\frac{s}{2}+\frac{1}{2}-\alpha}\xi\|^2_{V_n^0(\Omega)} + \frac{1}{2}\|I_\omega\Pi A_\Pi^{\frac{s}{2}+\frac{1}{2}-\alpha}\xi\|^2_{\mathbf{L}^2(\Omega)} \quad \forall\xi \in V_0^{1+s}(\Omega),
\end{aligned}$$

from which we deduce (4.12). □

Lemma 4.5. *For all $s \in [\frac{d-2}{2}, 1]$, the following estimate holds:*

$$(N(\xi)|\xi)_{\Pi,s} \leq C_s \|\xi\|_{\Pi,s}(A_\Pi \xi|\xi)_{\Pi,s} \quad \forall \xi \in V_0^{1+s}(\Omega). \tag{4.13}$$

Proof. If $s > 0$, from (2.1) with $(s_1, s_2, s_3) = (s, s, 1-s)$ and from (4.9) with $\theta = \frac{1}{2}$, we obtain

$$(N(\xi)|\xi)_{\Pi,s} = |\langle N(\xi)|\Pi^{(s)}\xi\rangle_{V_0^{s-1}(\Omega),V_0^{1-s}(\Omega)}| \leq C_s \|\xi\|_{V_0^s(\Omega)}\|\xi\|^2_{V_0^{1+s}(\Omega)}, \tag{4.14}$$

and (4.13) follows from (4.11) and (4.12). To treat the case $s = 0$, we first invoke (2.1) with $(s_1, s_2, s_3) = (\frac{1}{2}, 0, \frac{1}{2})$ and (4.9) with $(s, \theta) = (0, \frac{1}{4})$ to obtain the estimate

$$(N(\xi)|\xi)_{\Pi,0} = |(N(\xi)|\Pi^{(0)}\xi)| \leq C\|\xi\|^2_{V_0^{1/2}(\Omega)}\|\xi\|_{V_0^1(\Omega)}.$$

Thus, the interpolation inequality $\|\cdot\|_{V_0^{1/2}(\Omega)} \leq C\|\cdot\|^{1/2}_{V_n^0(\Omega)}\|\cdot\|^{1/2}_{V_0^1(\Omega)}$ yields (4.14) with $s = 0$, and (4.13) follows from (4.11) and (4.12). □

We are now in position to prove the following local stabilization result.

Theorem 4.6. *Let $s \in [\frac{d-2}{2}, 1]$. There exist $c_0 > 0$ and $\mu_1 > 0$ such that, if $\delta \in (0, \mu_1)$ and $y_0 \in \mathcal{V}_\delta^s$, system (4.1) admits a unique solution y in the set $\mathcal{S}_\delta^s$, and there exist $C > 0$ and $\sigma > 0$ such that*

$$\|y(t)\|_{V_0^s(\Omega)} \leq C\|y_0\|_{V_0^s(\Omega)}e^{-\sigma t} \quad \forall t \geq 0. \tag{4.15}$$

The sets $\mathcal{V}_\delta^s$ and $\mathcal{S}_\delta^s$ are defined in (4.2) and (4.3).

Proof. Let $c_0 > 0$ and $\mu_0 > 0$ be the ones given in Theorem 4.1 and let $0 \leq \mu_1 \leq \mu_0$. For $\delta \in (0, \mu_1)$ and for an initial condition $y_0 \in \mathcal{V}_\delta^s$, we consider the solution $y \in \mathcal{S}_\delta^s$ to (4.1). Hence, by multiplying the first equation in (4.1) by $\Pi^{(s)}y(t)$ we obtain:

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|^2_{\Pi,s} + (A_\Pi y(t)|y(t))_{\Pi,s} = (N(y(t))|y(t))_{\Pi,s}. \tag{4.16}$$

Thus, from (4.13), we deduce the existence of $C_s > 0$ such that:

$$\frac{d}{dt}\|y(t)\|^2_{\Pi,s} + 2(1 - C_s\|y(t)\|_{\Pi,s})(A_\Pi y(t)|y(t))_{\Pi,s} \leq 0. \tag{4.17}$$

If we choose y_0 so that $\|y_0\|_{\Pi,s} < \frac{1}{2C_s}$, then the mapping $t \longmapsto \|y(t)\|_{\Pi,s}$ is a nonincreasing function with values less than $\frac{1}{2C_s}$. As a consequence, for $C_1 > 0$ and $\sigma > 0$ such that $\|\cdot\|_{V_0^s(\Omega)} \leq C_1\|\cdot\|_{\Pi,s}$ and $2\sigma\|\cdot\|_{\Pi,s}^2 \leq (A_\Pi \cdot |\cdot)_{\Pi,s}$, if we choose $\mu_1 = \min\big(\mu_0, \frac{1}{2C_1C_s}\big)$, then we have $\|y_0\|_{\Pi,s} \leq \mu_1 \leq \frac{1}{2C_s}$ and (4.17) yields:

$$\frac{d}{dt}\|y(t)\|_{\Pi,s}^2 + 2\sigma\|y(t)\|_{\Pi,s}^2 \leq 0 \quad \forall t \in (0,\infty).$$

Finally, (4.15) follows from (4.11). □

Proof of Theorem 2.6. Let $s \in [\frac{d-2}{2}, 1]$, and let $c_0 > 0$ and $\mu_1 > 0$ be the ones given in Theorem 4.6. For $\delta \in (0, \mu_1)$ and for $y_0 \in \mathscr{V}_\delta^s$, we consider the solution $y \in \mathscr{S}_\delta^s$ to (4.1), where $\mathscr{V}_\delta^s$ and $\mathscr{S}_\delta^s$ are defined in (4.2) and (4.3). From Proposition 2.1 where $f = -I_\omega\Pi y$, we deduce that the formulation (4.1) is equivalent to following system:

$$\begin{aligned}
&\partial_t y - \nu\Delta y + (y\cdot\nabla)z_s + (z_s\cdot\nabla)y + (y\cdot\nabla)y + \nabla p = -I_\omega\Pi y,\\
&\nabla\cdot y = 0 \text{ in } (0,T)\times\Omega, \quad y = 0 \text{ on } (0,T)\times\Gamma, \quad y(0) = y_0 \in V_0^s(\Omega),
\end{aligned}$$

for all $T > 0$. Then if we write $(z, r) = (z_s + y, r_s + p)$, the Theorem 4.6 ensures the existence and the uniqueness of a solution to (2.16)–(2.17) in $\{(z_s, r_s)\} + \mathscr{S}_\delta^s \times H^{-\frac{1}{2}+\frac{s}{2}}(0,+\infty; \mathscr{H}^s(\Omega))$. Notice that from the continuous embedding $W(0,+\infty; V_0^{s+1}(\Omega), V_0^{s-1}(\Omega)) \hookrightarrow H^{\frac{s}{2}+\frac{1}{2}}(0,+\infty; V_n^0(\Omega))$, we deduce that $\partial_t y \in H^{-\frac{1}{2}+\frac{s}{2}}(0,+\infty; V_n^0(\Omega))$. By checking each term in (4.18), we obtain

$$\begin{aligned}
&\nabla p \in H^{-\frac{1}{2}+\frac{s}{2}}(0,+\infty; V_n^0(\Omega)) + L^2(0,+\infty; \mathbf{H}^{s-1}(\Omega)) \quad \text{and}\\
&\quad p \in H^{-\frac{1}{2}+\frac{s}{2}}(0,+\infty; \mathscr{H}^s(\Omega)),
\end{aligned}$$

as well as the following estimate:

$$\begin{aligned}
\|p\|_{H^{-\frac{1}{2}+\frac{s}{2}}(0,+\infty;H^s(\Omega))} &\leq c_1\|\nabla p\|_{H^{-\frac{1}{2}+\frac{s}{2}}(0,+\infty;\mathbf{H}^{s-1}(\Omega))}\\
&\leq (\|\partial_t y\|_{H^{-\frac{1}{2}+\frac{s}{2}}(0,+\infty;V_n^0(\Omega))} + \|(y\cdot\nabla)y\|_{L^2(0,+\infty;\mathbf{H}^{s-1}(\Omega))})\\
&\quad + \|\nu\Delta y + (y\cdot\nabla)z_s + (z_s\cdot\nabla)y + I_\omega\Pi y\|_{L^2(0,+\infty;\mathbf{H}^{s-1}(\Omega))}\\
&\leq c_2(\|y\|_{W(0,+\infty;V_0^{s+1}(\Omega)V_0^{s-1}(\Omega))} + \|y\|^2_{W(0,+\infty;V_0^{s+1}(\Omega),V_0^{s-1}(\Omega))}),
\end{aligned}$$

where c_1 and c_2 are two positive constants. Finally, for c_0 given in (4.2), we can choose $c = c_0(\max(c_2, \sqrt{c_2}, 1))^{-1}$ in (2.18). □

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