LYAPUNOV FUNCTION AND LOCAL FEEDBACK BOUNDARY STABILIZATION OF THE NAVIER-STOKES EQUATIONS

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Abstract. We study the local exponential stabilization, near a given steady-state flow, of solutions of the Navier-Stokes equations in a bounded domain. The control is performed through a Dirichlet boundary condition. We apply a linear feedback controller, provided by a well-posed infinite dimensional Riccati equation. We give a characterization of the domain of the closed-loop operator which is obtained from the closed-loop linearized Navier-Stokes system. We give a class of initial data for which a Lyapunov function is obtained. For all $s \in [0, 1/2[$, the stabilization of the 2-D Navier-Stokes equations is proved for initial data in $\mathbf{H}^s(\Omega) \cap V_n^0(\Omega)$, where $V_n^0(\Omega)$ is the space in which the Stokes operator is defined. We also obtain a 3-D stabilization result but only for a very specific set of initial data.

Key words. Navier-Stokes equations, Feedback stabilization, Dirichlet boundary control, Lyapunov function, Riccati equation

AMS subject classifications. 76D05, 76D07, 76D55, 93B52, 93C20, 93D15, 35B40, 35Q30

1. Introduction. Let Ω be a bounded and connected domain in $\mathbb{R}^d$ for $d = 2$ or $d = 3$, with a boundary $\Gamma = \partial\Omega$ of class C^4 which is composed of N connected components $\Gamma_1, \dots, \Gamma_N$. A stationary motion of an incompressible fluid in Ω is described by a velocity field z_e and a pressure function r_e which obey:

$$-\nu\Delta z_e + (z_e \cdot \nabla)z_e + \nabla r_e = f, \quad \nabla \cdot z_e = 0 \text{ in } \Omega \quad \text{and} \quad z_e = v_b \text{ on } \Gamma. \tag{1.1}$$

In this setting, $\nu > 0$ is the viscosity coefficient, $f \in \mathbf{H}^1(\Omega)$ and $v_b \in \mathbf{H}^{5/2}(\Gamma)$ obeys $\int_{\Gamma_j} v_b \cdot n = 0$ for all $j = 1 \dots N$, where n denotes the unit normal vector to Γ, exterior to Ω. Notice that here and in the following, we write in bold the spaces of vector fields: $\mathbf{H}^1(\Omega) = (H^1(\Omega))^d$, $\mathbf{H}^{5/2}(\Gamma) = (H^{5/2}(\Gamma))^d$, etc. We recall that a solution to the stationary Navier-Stokes equations (1.1) is known to exist in $\mathbf{H}^3(\Omega) \times H^2(\Omega)/\mathbb{R}$ [15, Chap. VIII, Thm. 4.1 and Thm. 5.2].

If z_e is an unstable equilibrium state, and if we assume that at time $t = 0$ the velocity is equal to $z_0 \neq z_e$, then even if z_0 is close to z_e, the resulting unsteady velocity $z(t)$ when $t > 0$ will not necessarily stay close to z_e. In order that $z(t)$ go back to z_e as $t \to +\infty$, a possible approach consists in looking for a feedback controller which is localized on the boundary Γ. In [21, 5, 4, 6] the authors use the solution of a Riccati equation to construct a feedback law $F : \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\Gamma)$ which is such that the solution to:

$$\partial_t z - \nu\Delta z + (z \cdot \nabla)z + \nabla r = f, \quad \nabla \cdot z = 0 \text{ in } (0, \infty) \times \Omega, \quad z(0) = z_0, \tag{1.2}$$

$$z = v_b + F(z - z_e) \text{ on } (0, \infty) \times \Gamma, \tag{1.3}$$

obeys $z(t) \longrightarrow z_e$ when $t \longrightarrow +\infty$. However, although it seems reasonable that the decay of $z(t) - z_e$ corresponds to the decreasing of some energy, in [21, 4, 6] it is not explained how to construct a Lyapunov function for system (1.2)-(1.3). Moreover, the Lyapunov function provided in [5] is obtained from an ill-posed Riccati equation. Hence, the main goal of the present paper is to construct a Lyapunov function for

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system (1.2)-(1.3) when F is a Riccati-type feedback law obtained from a well-posed Riccati equation.

In order to study the behaviour of the solution to (1.2)-(1.3) along the stationary trajectory (z_e, r_e), it is better to write $z_0 = z_e + y_0$ and $(z, r) = (z_e + y, r_e + p)$ where y_0 is the new initial datum and (y, p) is the new state variable. An easy formal calculation shows that $(y, p) = (z - z_e, r - r_e)$ obeys:

$$\partial_t y - \nu\Delta y + (y \cdot \nabla)z_e + (z_e \cdot \nabla)y + (y \cdot \nabla)y + \nabla p = 0 \ \text{ in } (0,\infty) \times \Omega, \tag{1.4}$$
$$\nabla \cdot y = 0 \text{ in } (0,\infty) \times \Omega, \quad y = Fy \text{ on } (0,\infty) \times \Gamma, \quad y(0) = y_0. \tag{1.5}$$

The key steps of the Riccati approach are the following ones. We linearize first system (1.4)-(1.5) around zero then we apply an unknown boundary control $u \in L^2((0,\infty)\times\Gamma)$ to the resulting linear system:

$$\partial_t y - \nu\Delta y + (y \cdot \nabla)z_e + (z_e \cdot \nabla)y + \nabla p = 0 \ \text{ in } (0,\infty) \times \Omega, \tag{1.6}$$
$$\nabla \cdot y = 0 \text{ in } (0,\infty) \times \Omega, \quad y = u \text{ on } (0,\infty) \times \Gamma, \quad y(0) = y_0. \tag{1.7}$$

Thus, to fit the framework of [18, Chap. 2], we introduce a Hilbert space of initial data H and we reduce system (1.6)-(1.7) to a dynamical system written in the abstract form:

$$y' + \mathcal{A}y = \mathcal{B}u \in \mathcal{D}(\mathcal{A}^*)', \quad y(0) = y_0 \in H, \tag{1.8}$$

where $\mathcal{A} : H \to \mathcal{D}(\mathcal{A}^*)'$ is the extension by transposition of an original unbounded operator still denoted by $\mathcal{A}$, with domain $\mathcal{D}(\mathcal{A})$ in H [18, 0.3], and where $\mathcal{B} : U \to \mathcal{D}(\mathcal{A}^*)'$ is a linear input operator defined on a Hilbert space of control U. We will explain later on how it can be achieved with the theory of [23]. Next, we fix an observation space Z and an operator $\mathcal{C} : H \to Z$, with a not too hight degree of unboundedness and so that the pair $(-\mathcal{A}, \mathcal{C})$ satisfies a detectablility condition [18, Chap. 2, (H-5) and (H-3')]. By this way, we obtain a feedback law

$$F = -\mathcal{B}^*\Pi \ : H \longrightarrow U, \tag{1.9}$$

from a bounded self-adjoint and nonnegative operator Π on H, which defines the value function of a minimizing problem associated with $\mathcal{C}$:

$$\big(\Pi y_0 | y_0\big)_H = \inf\left\{ \int_0^\infty \|\mathcal{C}y(t)\|_Z^2 dt + \int_0^\infty \|u(t)\|_U^2 dt \,\middle|\, (y,u) \text{ obeys } (1.8)\right\}. \tag{1.10}$$

Notice that the well-posedness of (1.10) relies on an adequate finite cost condition related to $\mathcal{C}$ [18, Chap.2, (2.1.12)]. Moreover, the optimal pair solution to (1.10) obeys the feedback relation $u = -\mathcal{B}^*\Pi y$. The optimal state is then the solution to (1.8) with the feedback control obtained from (1.9):

$$y' + \mathcal{A}y = -\mathcal{B}(\mathcal{B}^*\Pi)y, \quad y(0) = y_0 \in H. \tag{1.11}$$

The major interest of this approach is that the value function $(\Pi \cdot |\cdot)_H$ defined by (1.10) is the natural Lyapunov function of the closed-loop system (1.11). Indeed, it can be proved that Π satisfies the Riccati equation:

$$(\Pi\xi|\mathcal{A}\zeta)_H + (\mathcal{A}\xi|\Pi\zeta)_H + (\mathcal{B}^*\Pi\xi|\mathcal{B}^*\Pi\zeta)_U = (\mathcal{C}\xi|\mathcal{C}\zeta)_Z \quad \forall(\xi,\zeta) \in H \times H, \tag{1.12}$$

and if we multiply the first equation in (1.11) by Πy, the use of (1.12) where we set $\xi = \zeta = y(t)$ gives:

$$\partial_t(\Pi y(t)|y(t))_H + \|\mathcal{C}y(t)\|_Z^2 + \|\mathcal{B}^*\Pi y(t)\|_U^2 = 0.$$

Notice that the validity of (1.12) for test functions ξ and ζ which only belong to H is proved in [5, App. B.4], see also Remark 8 below. Hence, in order to stabilize the Navier-Stokes system, the Riccati-based strategy consists in applying the linear feedback law (1.9) to the nonlinear system. If we assume that the closed-loop Navier-Stokes system (1.4)-(1.5) can be reduced to a system of type:

$$y' + \mathcal{A}y = -\mathcal{B}(\mathcal{B}^*\Pi)y + \mathcal{N}(y) \ , \quad y(0) = y_0, \tag{1.13}$$

where $\mathcal{N}(y)$ is obtained from the nonlinear term $(y \cdot \nabla)y$, then similarly as in the linear case, we can verify (at least formally) that every solution to (1.13) obeys:

$$\partial_t(\Pi y(t)|y(t))_H + \|\mathcal{C}y(t)\|_Z^2 + \|\mathcal{B}^*\Pi y(t)\|_U^2 = 2(\mathcal{N}(y(t))|\Pi y(t))_H. \tag{1.14}$$

A brief glance at (1.14) suggests that, if we now choose $\mathcal{C}$ adequately unbounded in H, then the observation term $\|\mathcal{C}y(t)\|_Z^2$ may dominate the term $2(\mathcal{N}(y(t))|\Pi y(t))_H$ and may force local stabilization of y around zero. Hence, a first approach consists in choosing $\mathcal{C}$ unbounded enough so that the value function (1.10) is a Lyapunov function for system (1.13). The operator Π related to such unbounded observation is called a "hight-gain" Riccati operator [4, 6]. This strategy has been first used in [3, 7] to obtain a stabilization result for the 3-D Navier-Stokes system by means of a feedback control localized in an open subset of the domain. A "hight-gain" Riccati operator is also used in [5] to obtain a stabilization result for the 3-D Navier-Stokes system by means of a tangential feedback control localized on the boundary of the domain. The authors use an operator $\mathcal{C}$ which is the canonical isomorphism of $\mathbf{H}^{3/2+\epsilon}(\Omega)) \cap V_n^0(\Omega)$ onto $V_n^0(\Omega)$ for $\epsilon > 0$, where $V_n^0(\Omega)$ is the space in which the Stokes operator is defined. Such a choice ensures that the value function of the minimizing problem is a Lyapunov function for the controlled Navier-Stokes system, but there is no constructive way to calculate Π. Indeed, as it is explained in [4, 22], the too high degree of unboundedness of $\mathcal{C}$ does not allow to define a Riccati equation in a classical sense.

However, in the 2-D case, the initial paper [21] and its revisited version in [4, 6] illustrate the fact that a stabilization result can also be obtained with a bounded observation. Such a "low-gain" strategy seems to be the most reasonable one because it provides a bounded feedback law obtained from a well-posed Riccati equation. However, the value function (1.10) is no longer a Lyapunov function for the closed-loop nonlinear system (1.13). The stabilization results of [21, 4, 6] are obtained with a fixed point method which guarantees existence and uniqueness of a stable solution in a sufficiently small ball centered at the origin, and without exhibiting a Lyapunov function. Notice that the 3-D case is treated in [22] with a time dependent Riccati-type feedback law.

Hence, the natural question which arises is: can we construct a Lyapunov function for (1.13) when Π is a "low-gain" Riccati operator? Let us sketch the idea which allows to give a positive answer. Assume for simplicity that $\mathcal{C} = I_H$ (identity in H) and consider the solution Π to the Riccati equation (1.12). We introduce the free dynamic operator $\mathcal{A}_\Pi = \mathcal{A} + \mathcal{B}\mathcal{B}^*\Pi$ of the closed-loop system (1.11), and for $s \geq 0$ we construct the operator:

$$\Pi^{(s)} = \mathcal{A}_\Pi^{*1/2+s/2}\Pi\mathcal{A}_\Pi^{1/2+s/2}, \tag{1.15}$$

from which we can define the function:

$$\langle \Pi^{(s)} \cdot | \cdot \rangle = (\Pi \mathcal{A}_\Pi^{1/2+s/2} \cdot | \mathcal{A}_\Pi^{1/2+s/2} \cdot)_H. \qquad (1.16)$$

Notice that for the moment, (1.15) and (1.16) are only formal definitions. We will prove later on that (1.15) defines a bounded operator from $\mathcal{D}(\mathcal{A}_\Pi^{s/2})$ into $\mathcal{D}(\mathcal{A}_\Pi^{s/2})'$ and that the square root of (1.16) defines an norm on $\mathcal{D}(\mathcal{A}_\Pi^{s/2})$. If we now multiply the first equation in (1.13) by $\Pi^{(s)} y$, the use of (1.12) where we set $\xi = \zeta = \mathcal{A}_\Pi^{1/2+s/2} y(t)$ ensures that every solution to (1.13) obeys (at least formally):

$$\partial_t \langle \Pi^{(s)} y(t) | y(t) \rangle + \| \mathcal{A}_\Pi^{1/2+s/2} y(t) \|_H^2 \leq 2(\mathcal{N}(y(t)) | \Pi^{(s)} y(t))_H.$$

The above inequation suggests to seek an adequate $s \geq 0$ so that the term $\|\mathcal{A}_\Pi^{1/2+s/2} y(t)\|_H^2$ dominates the term $2(\mathcal{N}(y(t))|\Pi^{(s)} y(t))_H$, which is to say that $\langle \Pi^{(s)} \cdot | \cdot \rangle$ is a Lyapunov function of system (1.13). However, this function is now only well-defined on the domain of a fractional power of $\mathcal{A}_\Pi$: that is the price to pay to obtain a Lyapunov function for system (1.13) with such a "low-gain" strategy. More precisely, because it can be proved that $\sqrt{\langle \Pi^{(s)} \cdot | \cdot \rangle}$ defines a norm equivalent to the one of $\mathcal{D}(\mathcal{A}_\Pi^{s/2})$, to obtain a continuous mapping $t \longmapsto \langle \Pi^{(s)} y(t) | y(t) \rangle$ we shall assume that y_0 belongs to $\mathcal{D}(\mathcal{A}_\Pi^{s/2})$. Then, we need a precise characterization of the space $\mathcal{D}(\mathcal{A}_\Pi^{s/2})$. Notice that in the case of a distributed Riccati-type feedback control, the same approach is used in [1] to construct a large class of Lyapunov functions.

The paper is organized as follows.

In Section **2**, we give the notations used throughout the paper, we give an abstract formulation for the Oseen and Navier-Stokes systems, and we state our main theorems. In particular, to reduce (1.6)-(1.7) to a system of the form (1.8) we introduce the state space $H = V_n^0(\Omega) = \{y \in \mathbf{L}^2(\Omega) \mid \nabla \cdot y = 0 \text{ in } \Omega,\ y \cdot n = 0 \text{ on } \Gamma\}$ and the control space $U = V^0(\Gamma) = \{y \in \mathbf{L}^2(\Gamma) \mid \int_\Gamma y \cdot n = 0\}$, and we recall the theory of [23] which allows to rewrite system (1.6)-(1.7) in the following abstact form:

$$Py' + APy + A_e Py = Bu \ \in \mathcal{D}((A + A_e)^*)', \quad Py(0) = Py_0 \in V_n^0(\Omega), \quad (1.17)$$

$$(I - P)y = (I - P)Du. \qquad (1.18)$$

In the above setting, P is the orthogonal projector from $\mathbf{L}^2(\Omega)$ onto $V_n^0(\Omega)$, A is the Stokes operator, A_e is the operator associated with the convective term $(y \cdot \nabla) z_e + (z_e \cdot \nabla) y$, D is an adequate lifting operator and $B = (\lambda_0 + A + A_e)PD : V^0(\Gamma) \to \mathcal{D}((A + A_e)^*)'$. Moreover, since in (1.17)-(1.18) one observes that the velocity y is entirely determined by its projected part Py and by the control u, system (1.6)-(1.7) can be reduced to:

$$y' + (A + A_e) y = Bu \ \in \mathcal{D}((A + A_e)^*)', \quad y(0) = y_0 \in V_n^0(\Omega), \qquad (1.19)$$

where y in (1.19) plays the role of Py in (1.17).

In Section **3**, we obtain a Riccati operator Π from the value function of the minimizing problem:

$$\inf \left\{ \int_0^\infty \|y(t)\|_{V_n^0(\Omega)}^2 dt + \int_0^\infty \|u(t)\|_{V^0(\Gamma)}^2 dt \ \middle| \ (y, u) \text{ obeys (1.19)} \right\},$$

whose optimal state obeys the following closed-loop system:

$$y' + A_\Pi y = 0 \text{ and } y(0) = y_0 \in V_n^0(\Omega), \text{ where } A_\Pi = A + A_e + B(B^* \Pi). \qquad (1.20)$$

In Section **4**, for a value $s \in [0,1]$ we prove that $\Pi^{(s)} = A_\Pi^{*1/2+s/2}\Pi A_\Pi^{1/2+s/2}$ defines a bounded operator from $\mathcal{D}(A_\Pi^{s/2})$ into $\mathcal{D}(A_\Pi^{s/2})'$, we introduce the following new inner product on $\mathcal{D}(A_\Pi^{s/2})$:

$$\big(\xi|\zeta\big)_{\Pi,s} = \langle \Pi^{(s)}\xi|\zeta\rangle_{\mathcal{D}(A_\Pi^{s/2})',\mathcal{D}(A_\Pi^{s/2})},$$

and when $y_0 \in \mathcal{D}(A_\Pi^{s/2})$, we prove that the function $\mathcal{V}_s(\cdot)$ defined by $\mathcal{V}_s(\xi) = (\xi|\xi)_{\Pi,s}$ is a Lyapunov function for the linear closed-loop system (1.20).

In Section **5**, we prove that $\mathcal{D}({A_\Pi}^{s/2})$ for $s \in [0,2]$ is a closed subspace of $V_n^0(\Omega)$ which is equipped with the same topology as $\mathbf{H}^s(\Omega)$, and which involves a very specific trace condition when $s \geq 1/2$. The proof is based on the characterization of $\mathcal{D}(A_\Pi)$ and on the equality $\mathcal{D}({A_\Pi}^{s/2}) = [\mathcal{D}(A_\Pi), V_n^0(\Omega)]_{1-s/2}$. One shall underline that the assumption Ω of class C^4 is used to identify $\mathcal{D}(A_\Pi)$.

In Section **6**, we study the well-posedness of system (1.2)-(1.3) for $F = -(B^*\Pi)P$, which can be reduced to:

$$y' + A_\Pi y + N_\Pi(y) = 0, \quad y(0) = y_0, \tag{1.21}$$

where $N_\Pi(\cdot)$ is an adequate nonlinear operator. By combining a fixed point method and a priori estimates obtained from the $(\cdot|\cdot)_{\Pi,s}-$product of (1.21) with y, we prove that for $s \in [\frac{d-2}{2}, 1]$ and y_0 small enough in $\mathcal{D}(A_\Pi^{s/2})$, system (1.21) admits a solution which is exponentially stable and which is unique within the class of functions belonging to $L^\infty(0,T;\mathcal{D}(A_\Pi^{s/2})) \cap L^2(0,T;\mathcal{D}(A_\Pi^{s/2+1/2}))$ for all $T > 0$. In particular, because for $s \in [0,1/2[$ we have $\mathcal{D}(A_\Pi^{s/2}) = \mathbf{H}^s(\Omega) \cap V_n^0(\Omega)$, we obtain a 2-D stabilization result generalizing the one of [21] when $s \in [0,1/4]$. However, when $d = 3$ we must assume $s \geq 1/2$, and because a trace condition appears in the definition of $\mathcal{D}(A_\Pi^{s/2})$ we only obtain a 3-D stabilization result for a very specific set of initial data. We shall underline that the minimal Sobolev index $s = \frac{d-2}{2}$ is sharp in the sense that it is the minimal value for which the first equality in (1.21) make sense in $L^2(0,T;\mathcal{D}(A_\Pi^{*s/2+1/2})')$ when y' and $A_\Pi y$ belongs to $L^2(0,T;\mathcal{D}(A_\Pi^{*s/2+1/2})')$, or saying it differently, it is the minimal value for which a fixed point solution can be obtained in $L^2(0,T;\mathcal{D}(A_\Pi^{s/2+1/2})) \cap H^1(0,T;\mathcal{D}(A_\Pi^{*s/2+1/2})')$. One shall also refer to the introduction of [2] where the sharpness of $s = 1/2$ when $d = 3$ is discussed by directly considering equation (1.4).

Finally, Section **7** deals with a boundary control which is localized in a part of Γ.

2. Preliminaries and main results.

2.1. Notations. Throughout the following, if Z is a Banach space, we denote by $\|\cdot\|_Z$ its corresponding norm, we denote by Z' its dual space and by $\langle\cdot|\cdot\rangle_{Z',Z}$ the Z'-Z duality pairing. We use the notation $Z_1 \hookrightarrow Z_2$ to say that a space Z_1 is continuously embedded into Z_2. We denote by $\mathcal{L}(Z_1, Z_2)$ the space of all bounded operators from Z_1 into Z_2 and we use the shorter expression $\mathcal{L}(Z) = \mathcal{L}(Z,Z)$. The domain of a closed linear mapping A in Z is denoted by $\mathcal{D}(A)$, and A^* denotes the adjoint of A.

For a Hilbert space X and for $0 < T \leq \infty$, the space $L^2(0,T;X)$ is the usual vector-valued Lebesgue space, $H^s(0,T;X)$ for $s \geq 0$ is the usual vector-valued Sobolev space [16], $H_0^s(0,T;X)$ denotes the closure in $H^s(0,T;X)$ of the space of infinitely differentiable and compactly supported functions of $t \in]0,T[$ with values in X, and $H^{-s}(0,T;X')$ denotes the dual space of $H_0^s(0,T;X)$. When $T = +\infty$ we use the shorter expressions $L^2(X) = L^2(0,+\infty;X)$, $H^s(X) = H^s(0,+\infty;X)$, etc. Moreover, $L^\infty(X)$ (resp. $C_b(X)$) is the space of bounded (resp. continuous and bounded) functions of $t \in [0,\infty[$ with values in X. For two Hilbert spaces X, Y and for $0 < T \leq \infty$ we also define:

$$W(0,T;X,Y) = L^2(0,T;X) \cap H^1(0,T;Y),$$

and we use the shorter expression $W(X,Y) = W(0,+\infty;X,Y)$. It is well known that if $X \hookrightarrow Y$, then $W(X,Y)$ is continuously embedded in $C_b([X,Y]_{1/2})$ [25, 1.8 (2), p.44 and Rem.3 p.143]. We recall that the space $[X,Y]_\theta$ for $\theta \in [0,1]$ denotes the interpolation space obtained from X and Y with the complex interpolation method [25]. Finally, we denote by $L^2_{loc}(X)$ the space of functions belonging to $L^2(0,T;X)$ for all $T > 0$, and we define $L^\infty_{loc}(X)$, $H^s_{loc}(X)$ etc, analogously.

The letter C denotes a generic positive constant that may change from line to line. When particular positive constants are needed, we use a subscript or small character: C_0, C_1, C_2, etc or c_0, c_1, c_2 etc.

Throughout the following, Ω is a bounded and connected domain in $\mathbb{R}^d$ for $d = 2$ or $d = 3$, with a boundary $\Gamma = \partial\Omega$ of class C^4 which is composed of N connected components $\Gamma_1, \ldots, \Gamma_N$. By $L^2(\Omega)$, $L^2(\Gamma)$, $H^s(\Omega)$, $H^s(\Gamma)$, $H^s_0(\Omega)$ and $H^{-s}(\Omega) = (H^s_0(\Omega))'$ for $s \geq 0$, we denote the usual Lebesgue and Sobolev spaces of scalar functions in Ω or in Γ, and we write in bold the spaces of vector-valued functions: $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$, $\mathbf{L}^2(\Gamma) = (L^2(\Gamma))^d$, etc. Moreover, $(\cdot|\cdot)$ is the usual inner product in $\mathbf{L}^2(\Omega)$ and $(\cdot|\cdot)_\Gamma$ is the usual inner product in $\mathbf{L}^2(\Gamma)$. We also introduce different spaces of free divergence functions and some corresponding trace spaces:

$$\begin{aligned}
V^s(\Omega) &= \Big\{ y \in \mathbf{H}^s(\Omega) \mid \nabla \cdot y = 0 \text{ in } \Omega \Big\}, \quad s \geq 0, \\
V^s_n(\Omega) &= \Big\{ y \in \mathbf{H}^s(\Omega) \mid \nabla \cdot y = 0 \text{ in } \Omega,\ y \cdot n = 0 \text{ on } \Gamma \Big\}, \quad s \geq 0, \\
V^s(\Gamma) &= \Big\{ y \in \mathbf{H}^s(\Gamma) \mid \int_\Gamma y \cdot n = 0 \Big\}, \quad s \geq 0.
\end{aligned}$$

In the above setting, n denote the unit normal to Γ outward Ω. We recall that if $y \in \mathbf{L}^2(\Omega)$ obeys $\nabla \cdot y \in L^2(\Omega)$ then its normal trace $y \cdot n$ is well-defined in $H^{-1/2}(\Gamma)$ [14, III. 3].

We denote by Δ the vector-valued Laplace operator in $\mathbf{L}^2(\Omega)$ with domain $\mathcal{D}(\Delta) = \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega)$, and we define:

$$V^s_0(\Omega) = \mathcal{D}((-\Delta)^{s/2}) \cap V^0_n(\Omega) \quad \text{for all} \quad s \in [0,2].$$

Because the self-adjointness of Δ guarantees $\mathcal{D}((-\Delta)^{s/2}) = [\mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega), \mathbf{L}^2(\Omega)]_{1-s/2}$ for all $s \in [0,2]$ [8, Chap. 1, Cor. 6.1], from [16, Thm. 8.1] we obtain the following equalities:

$$\begin{aligned}
V^s_0(\Omega) &= V^s_n(\Omega), \quad s \in [0,1/2[, \\
V^{1/2}_0(\Omega) &= \Big\{ y \in V^{1/2}_n(\Omega) \ \mid \ y \in \mathbf{L}^2_{-1/2}(\Omega) \Big\}, \\
V^s_0(\Omega) &= \Big\{ y \in V^s_n(\Omega) \ \mid \ y = 0 \ \text{ on } \ \Gamma \Big\}, \quad s \in]1/2,2],
\end{aligned}$$

where $\mathbf{L}^2_{-1/2}(\Omega)$ is the space of functions $y \in \mathbf{L}^2(\Omega)$ such that $\int_\Omega \rho(x)^{-1}|y|^2 < +\infty$ and $\rho(x)$ is the distance from x to Γ. Notice that the subscript 0 in $V^s_0(\Omega)$ only means that one may have a vanishing Dirichlet boundary condition. Finally, for $s > 2$ we define $V_0^s(\Omega) = V_0^2(\Omega) \cap \mathbf{H}^s(\Omega)$ and for $s < 0$ we define $V_0^s(\Omega) = (V_0^{-s}(\Omega))'$, the dual space of $V_0^{-s}(\Omega)$ with respect to the pivot space $V_n^0(\Omega)$.

We also need the so-called Leray projector $P \in \mathcal{L}(\mathbf{L}^2(\Omega), V_n^0(\Omega))$ which is the orthogonal projector from $\mathbf{L}^2(\Omega)$ onto $V_n^0(\Omega)$ [14, Chap. III, Thm. 1.1]. Since Ω is of class C^4, by studying its related Neumann problem [14, Chap.III, Lem 1.2], we obtain that P obeys the following regularity property:

$$P \in \mathcal{L}(\mathbf{H}^s(\Omega), V_n^s(\Omega)) \quad \forall s \in [0, 3]. \tag{2.1}$$

Notice that P can also be extended to a bounded linear operator from $\mathbf{H}^{-1}(\Omega)$ onto $V_0^{-1}(\Omega)$ [5, App. A].

We define the spaces of pressures with free mean:

$$L_0^2(\Omega) = \left\{ p \in L^2(\Omega) \mid \int_\Omega p = 0 \right\} \quad \text{and} \quad \mathcal{H}^s(\Omega) = H^s(\Omega) \cap L_0^2(\Omega), \qquad s \geq 0.$$

Finally, we set $Q = (0, +\infty) \times \Omega$ and $\Sigma = (0, +\infty) \times \Gamma$, and for $s \geq 0$ we define:

$$V^{s,s/2}(Q) = L^2(V^s(\Omega)) \cap H^{s/2}(V^0(\Omega)),$$
$$V^{s,s/2}(\Sigma) = L^2(V^s(\Gamma)) \cap H^{s/2}(V^0(\Gamma)).$$

We denote by $V_{loc}^{s,s/2}(Q)$ the space of functions belonging to $L^2(0, T; V^s(\Omega)) \cap H^{s/2}(0, T; V^0(\Omega))$ for all $T > 0$.

2.2. Navier-Stokes and Oseen system with a Dirichlet boundary condition. This subsection is devoted to the abstract formulation of Navier-Stokes and Oseen systems with a Dirichlet boundary condition.

First, we define the Stokes operator in $V_n^0(\Omega)$ by:

$$\mathcal{D}(A) = V_0^2(\Omega), \quad Ay = -\nu P \Delta y.$$

It is well known that $(\mathcal{D}(A), A)$ is nonnegative self-adjoint and positive definite, that its fractional powers satisfy $\mathcal{D}(A^{s/2}) = V_0^s(\Omega)$ for all $s \in [0, 2]$ [13], and that it is the infinitesimal generator of an analytic semigroup $(e^{-At})_{t>0}$ on $V_n^0(\Omega)$.

Thus, we introduce the linear operator

$$\mathcal{D}(A_e) = V_0^1(\Omega), \quad A_e y = P(y \cdot \nabla) z_e + P(z_e \cdot \nabla) y,$$

and we define the so-called Oseen operator $A + A_e$ with domain $\mathcal{D}(A + A_e) = \mathcal{D}(A) = V_0^2(\Omega)$. Since it can be viewed as a perturbation of A with a perturbation term A_e with domain $\mathcal{D}(A_e) = \mathcal{D}(A^{1/2})$, we obtain the analyticity of $(e^{-(A+A_e)t})_{t>0}$ on $V_n^0(\Omega)$ from the analyticity of $(e^{-At})_{t>0}$ on $V_n^0(\Omega)$ [20, Chap.3, Cor.2.4]. The adjoint of $(\mathcal{D}(A + A_e), A + A_e)$ is defined by $\mathcal{D}((A + A_e)^*) = V_0^2(\Omega)$ and $(A + A_e)^* = A + A_{e,T}$ where

$$\mathcal{D}(A_{e,T}) = V_0^1(\Omega), \quad A_{e,T} y = P(\nabla z_e)^T y - P(z_e \cdot \nabla) y.$$

In the above setting the superscript "^{T}" denotes the transpose. Moreover, we can choose $\lambda_0 > 0$ such that

$$\langle (\lambda_0 + A + A_e) y | y \rangle_{V_0^{-1}(\Omega), V_0^1(\Omega)} \geq \frac{\nu}{2} \|y\|_{V_0^1(\Omega)}^2 \quad \forall y \in V_0^1(\Omega),$$

and we have:

$$\mathcal{D}((\lambda_0 + A + A_e)^\theta) = \mathcal{D}((\lambda_0 + A + A_{e,T})^\theta) = V_0^{2\theta}(\Omega) \quad \forall \theta \in [0,1],$$

see for instance [23, Lem. 4.1]. As a consequence, for all $\theta \in [0,1]$ and $T > 0$, optimal regularity results [8, Chap. 3, Par. 2] ensure that the mapping below is an isomorphism:

$$\begin{array}{ccc} W(0,T;V_0^{2\theta}(\Omega), V_0^{2(\theta-1)}(\Omega)) & \longrightarrow & L^2(0,T;V_0^{2(\theta-1)}(\Omega)) \times V_0^{2\theta-1}(\Omega)), \\ y & \longmapsto & (\ y' + Ay + A_e y\ ,\ y(0)\ ). \end{array} \tag{2.2}$$

Next, we introduce the Dirichlet operator $D : V^0(\Gamma) \longrightarrow V^0(\Omega)$ associated with $\lambda_0 + A + A_e$. For $u \in V^0(\Gamma)$ the function $w = Du$ is defined by:

$$\lambda_0 w - \nu \Delta w + (w \cdot \nabla) z_e + (z_e \cdot \nabla) w + \nabla q = 0, \quad \nabla \cdot w = 0, \quad w|_\Gamma = u.$$

About such a Dirichlet operator, see [23, App. 2].

Proposition 1. *(i) The operator D is bounded from $V^0(\Gamma)$ into $V^0(\Omega)$ and it satisfies:*

$$D \in \mathcal{L}(V^{s-1/2}(\Gamma)), V^s(\Omega)) \quad \textit{for all} \quad s \in [0,2]. \tag{2.3}$$

(ii) The operator $D^* \in \mathcal{L}(V^0(\Omega), V^0(\Gamma))$, the adjoint of D, is defined by

$$D^* f = rn - \nu \partial_n \varphi,$$

where $(\varphi, r) \in V_0^2(\Omega) \times H^1(\Omega)$ is the unique pair solution to:

$$\lambda_0 \varphi - \nu \Delta \varphi + (\nabla z_e)^T \varphi - (z_e \cdot \nabla) \varphi + \nabla r = f,\ \nabla \cdot \varphi = 0,\ \varphi|_\Gamma = 0,\ \int_\Gamma r = 0.$$

Moreover, the operator D^* obeys:

$$D^* \in \mathcal{L}(V_0^s(\Omega), V^{s+1/2}(\Gamma)) \quad \forall s \in [0,2].$$

Proof. See [23, App. 2].

We are now in position to define the input operator B appearing in (1.17) and in (1.19).

Definition 1. *Let us define $B \in \mathcal{L}(V^0(\Gamma); V_0^{-2}(\Omega))$ as follows:*

$$\langle Bu|\varphi\rangle_{V_0^{-2}(\Omega), V_0^2(\Omega)} = (PDu|(\lambda_0 + A + A_e)\varphi) \quad \forall (u, \varphi) \in V^0(\Gamma) \times V_0^2(\Omega). \tag{2.4}$$

Remark 1. *With the transposition method, $A + A_e$ can be extended to an operator $\widetilde{A + A_e}$ defined in $\mathcal{D}(A + A_{e,T})'$ with domain $\mathcal{D}(\widetilde{A + A_e}) = V_n^0(\Omega)$ in the following way:*

$$\langle (\widetilde{A + A_e}) v|\varphi\rangle_{\mathcal{D}(A+A_{e,T})', \mathcal{D}(A+A_{e,T})} = (v|(A + A_{e,T})\varphi) \quad \forall (v, \varphi) \in V_n^0(\Omega) \times \mathcal{D}(A + A_{e,T}).$$

The space $\mathcal{D}(A + A_{e,T})'$, which is the dual space of $\mathcal{D}(A + A_{e,T})$ with respect to the pivot space $V_n^0(\Omega)$, is called the extrapolation space generated by $A + A_e$ [18, 0.3]. Hence, an equivalent definition of (2.4) is $B = (\lambda_0 + \widetilde{A + A_e}) PD$. Throughout the following, *for readable convenience, we shall continue to denote by* $A+A_e$ *the extension* $\widetilde{A+A_e}$ *and we shall write* $B=(\lambda_0+A+A_e)PD$.

PROPOSITION 2. *For all* $\varepsilon\in]0,1/4[$, *the linear operator* B *obeys:*

$$(\lambda_0+A+A_e)^{-3/4-\varepsilon}B\in\mathcal{L}(V^0(\Gamma),V^0(\Omega)).$$

Proof. It is a direct consequence of equality $(\lambda_0+A+A_e)^{-3/4-\varepsilon}B=(\lambda_0+A+A_e)^{1/4-\varepsilon}PD$ with $\mathcal{D}((\lambda_0+A+A_e)^{1/4-\varepsilon})=V_n^{1/2-2\varepsilon}(\Omega)$, $P\in\mathcal{L}(\mathbf{H}^{1/2-2\varepsilon}(\Omega),V_n^{1/2-2\varepsilon}(\Omega))$ and $D\in\mathcal{L}(V^0(\Gamma),V_n^{1/2-2\varepsilon}(\Omega))$.

PROPOSITION 3. *The adjoint of* B *is* $B^*\in\mathcal{L}(V_0^2(\Omega),V^0(\Gamma))$ *defined by:*

$$B^*\varphi=rn-\nu\partial_n\varphi\quad\forall\varphi\in V_0^2(\Omega),\tag{2.5}$$

where $r\in H^1(\Omega)$ *is the unique solution to the Neumann problem:*

$$\Delta r=\nabla\cdot(z_e\cdot\nabla-(\nabla z_e)^T)\varphi\ \ \textit{in}\ \Omega,\qquad\int_\Gamma r=0,$$
$$\partial_n r=(\nu\Delta-(\nabla z_e)^T+z_e\cdot\nabla)\varphi\cdot n\ \ \textit{on}\ \Gamma.$$

Moreover, the operator B^* *obeys:*

$$B^*\in\mathcal{L}(V_0^{s+3/2}(\Omega),V^s(\Gamma))\quad\forall s\in\Big]0,3/2\Big].\tag{2.6}$$

Proof. To prove (2.5) we first observe that we have $B^*=D^*(\lambda_0+A+A_{e,T})\in\mathcal{L}(V_0^2(\Omega),V^0(\Gamma))$ from $B=(\lambda_0+A+A_e)PD\in\mathcal{L}(V^0(\Gamma),V_0^{-2}(\Omega))$. Hence, according to the definition of D^*, for $\varphi\in V_0^2(\Omega)$ we have $B^*\varphi=rn-\nu\partial_n\widetilde{\varphi}$, where $(\widetilde{\varphi},r)\in V_0^2(\Omega)\times H^1(\Omega)$ satisfies:

$$\lambda_0\widetilde{\varphi}-\nu\Delta\widetilde{\varphi}+(\nabla z_e)^T\widetilde{\varphi}-(z_e\cdot\nabla)\widetilde{\varphi}+\nabla r=(\lambda_0+A+A_{e,T})\varphi\ \text{ and }\ \int_\Gamma r=0.$$

Thus, by successively applying P and $I-P$ to the above equality we deduce that $\widetilde{\varphi}=\varphi$ and that

$$\nabla r=(I-P)(\nu\Delta\varphi-(\nabla z_e)^T\varphi+(z_e\cdot\nabla)\varphi),$$

which is equivalent to

$$\Delta r=\nabla\cdot(\nu\Delta\varphi-(\nabla z_e)^T\varphi+(z_e\cdot\nabla)\varphi)\ \text{ and }\ \partial_n r=(\nu\Delta\varphi-(\nabla z_e)^T\varphi+(z_e\cdot\nabla)\varphi)\cdot n\text{ on }\Gamma.$$

We conclude by remarking that $\nabla\cdot\Delta\varphi=0$. Finally, (2.6) is an easy consequence of $\partial_n\in\mathcal{L}(V_0^{s+3/2}(\Omega),V^s(\Gamma))$ for $s>0$ and of regularity results for the Laplace problem with a nonhomogeneous Neumann condition.

REMARK 2. *Because* φ *in (2.5) satisfies* $\nabla\cdot\varphi=0$ *and* $\varphi|_\Gamma=0$, *we deduce that its normal derivative* $\partial_n\varphi$ *is tangential [5, Lem. 3.3.1]. Hence,* rn *and* $-\nu\partial_n\varphi$ *are respectively the normal component and the tangential component of* $B^*\varphi$.

We are now in position to define the evolution Oseen system with a nonhomogeneous Dirichlet boundary condition. For an initial datum obeying $Py_0\in V_n^0(\Omega)$, and for a boundary value $u \in L^2(V^0(\Gamma))$, the weak formulation of (1.6)-(1.7) which is given in [23] is

$$\begin{aligned} Py' + APy + A_e Py = Bu \ \in \mathcal{D}(A + A_{e,T})', \quad Py(0) = Py_0, \\ (I - P)y = (I - P)Du. \end{aligned} \tag{2.7}$$

However, since the solution y is entirely determined by Py and u, the study of the Oseen system can be reduced to the study of the following linear system:

$$y' + Ay + A_e y = Bu \ \in \mathcal{D}(A + A_{e,T})', \quad y(0) = y_0 \in V_n^0(\Omega), \tag{2.8}$$

where y in (2.8) plays the role of Py in (2.7). Notice that because (2.2) for $\theta = 0$ is an isomorphism, the solution to (2.8) exists and is unique in $W_{loc}(V_n^0(\Omega), V_0^{-2}(\Omega))$.

Next, we introduce the following trilinear form:

$$b(v_1, v_2, v_3) = \int_\Omega (v_1 \cdot \nabla) v_2 \cdot v_3 \quad \forall (v_1, v_2, v_3) \in \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega),$$

which is known to satisfy the estimate

$$b(v_1, v_2, v_3) \leq C \|v_1\|_{\mathbf{H}^{s_1}(\Omega)} \|v_2\|_{\mathbf{H}^{1+s_2}(\Omega)} \|v_3\|_{\mathbf{H}^{s_3}(\Omega)}, \tag{2.9}$$

where $(v_1, v_2, v_3) \in \mathbf{H}^{s_1}(\Omega) \times \mathbf{H}^{1+s_2}(\Omega) \times \mathbf{H}^{s_3}(\Omega)$ and s_1, s_2 and s_3 are real nonnegative numbers such that $s_1 + s_2 + s_3 \geq \frac{d}{2}$ if $s_i \neq \frac{d}{2}$, $i = 1, 2, 3$ or $s_1 + s_2 + s_3 > \frac{d}{2}$ if $s_i = \frac{d}{2}$, for at least one i [9, Prop. 6.1, (6.10)]. Notice that for the real nonnegative numbers s_1, s_2 and s_3 given in (2.9), we also have the following estimate:

$$b(v_1, v_2, v_3) \leq C \|v_1\|_{V^{s_1}(\Omega)} \|v_2\|_{V^{s_2}(\Omega)} \|v_3\|_{\mathbf{H}_0^{1+s_3}(\Omega)}, \tag{2.10}$$

for $(v_1, v_2, v_3) \in V^{s_1}(\Omega) \times V^{s_2}(\Omega) \times \mathbf{H}_0^{1+s_3}(\Omega)$. Indeed, it is a direct consequence of (2.9), and of the following antisymetry property:

$$b(v_1, v_2, v_3) = -b(v_1, v_3, v_2) \quad \forall (v_1, v_2, v_3) \in V^1(\Omega) \times V^1(\Omega) \times \mathbf{H}_0^1(\Omega),$$

which is obtained from an integration by parts. Thus, we define the nonlinear mapping:

$$N : V^1(\Omega) \to V_0^{-1}(\Omega), \quad \langle N(y) | v \rangle_{V_0^{-1}(\Omega), V_0^1(\Omega)} = b(y, y, w), \tag{2.11}$$

which allows to state the following abstract formulation of Navier-Stokes system with a nonhomogeneous Dirichlet boundary condition:

$$Py' + APy + A_e Py + N(y) = Bu \ \in \mathcal{D}(A + A_{e,T})', \quad y(0) = y_0, \tag{2.12}$$

$$(I - P)y = (I - P)Du. \tag{2.13}$$

The following proposition gives an interpretation of (2.12)-(2.13) in terms of partial differential equations.

Proposition 4. *Let $(z_e, r_e) \in V^3(\Omega) \times \mathcal{H}^2(\Omega)$ be a solution to (1.1), let $s \in [0, 1]$, let $y_0 \in V^s(\Omega)$ and let $u \in L^2_{loc}(V^{1/2+s}(\Gamma))$. Then the following results hold.*
(i) If $y \in V_{loc}^{1+s, 1/2+s/2}(Q)$ obeys (2.12)-(2.13) then there is a unique pressure function $p \in H_{loc}^{-1/2+s/2}(\mathcal{H}^s(\Omega))$ such that $(z, r) = (z_e, r_e) + (y, p)$ satisfies:

$$\partial_t z - \nu \Delta z + (z \cdot \nabla) z + \nabla r = f, \quad \nabla \cdot z = 0 \ \text{in } Q, \tag{2.14}$$

$$z = u + v_b \ \text{on } \Sigma, \quad z(0) = z_e + y_0 \in V^s(\Omega). \tag{2.15}$$

(ii) Conversely, if (z,p) belongs to

$$\big\{(z_e,r_e)\big\} + V_{loc}^{1+s,1/2+s/2}(Q) \times H_{loc}^{-1/2+s/2}(\mathcal{H}^s(\Omega)),$$

and obeys (2.14)-(2.15), then $y = z - z_e$ satisfies (2.12)-(2.13).

Proof. Let us give a brief sketch of the proof of *(i)* and *(ii)*.
(i) If we assume that $y \in V_{loc}^{1+s,1/2+s/2}(Q)$ obeys (2.12)-(2.13), the same argument as in [2, Thm. 4.8] allows to recover the trace condition $y|_\Sigma = u$ from an integration by parts, and to deduce that y satisfies:

$$\frac{d}{dt}\int_\Omega y(t)\cdot\varphi + \nu\int_\Omega \big(\nabla y(t):\nabla\varphi + (y(t)\cdot\nabla)z_e\cdot\varphi + (z_e\cdot\nabla)y(t)\cdot\varphi\big) + \int_\Omega (y(t)\cdot\nabla)y(t)\cdot\varphi = 0,$$

for all $\varphi \in V_0^1(\Omega)$. Thus, we define $\mathcal{Y}(t) = \int_0^t y(\tau)d\tau$ and $\mathcal{N}(\mathcal{Y})(t) = \int_0^t N(y)(\tau)d\tau$, and by integrating in time the above equation we obtain:

$$\big\langle y(t) - y_0 - \nu\Delta\mathcal{Y}(t) + (\mathcal{Y}(t)\cdot\nabla)z_e + (z_e\cdot\nabla)\mathcal{Y}(t) + \mathcal{N}(\mathcal{Y})(t)|\varphi\big\rangle = 0 \quad \forall\varphi \in V_0^1(\Omega).$$

According to [24, Rem.1.4(i), Chap.1, p. 15], the above equality means that for almost each time $t > 0$ there is a unique $\mathcal{P}(t) \in L_0^2(\Omega)$ obeying:

$$\nabla\mathcal{P}(t) = y(t) - y_0 - \nu\Delta\mathcal{Y}(t) + (\mathcal{Y}(t)\cdot\nabla)z_e + (z_e\cdot\nabla)\mathcal{Y}(t) + \mathcal{N}(\mathcal{Y})(t). \tag{2.16}$$

Hence, by checking the regularity of each term at the right of the above equality, we deduce that $\nabla\mathcal{P} \in H_{loc}^{1/2+s/2}(\mathbf{H}^{s-1}(\Omega))$. As a consequence, $p = -\frac{d}{dt}\mathcal{P}$ belongs to $H_{loc}^{-1/2+s/2}(\mathcal{H}^s(\Omega))$ and by differentiating (2.16) we easily verify that (y,p) satisfies:

$$\begin{aligned} &\partial_t y - \nu\Delta y + (y\cdot\nabla)z_e + (z_e\cdot\nabla)y + (y\cdot\nabla)y + \nabla p = 0, \\ &\nabla\cdot y = 0 \text{ in } Q, \quad y = u \text{ on } \Sigma, \quad y(0) = y_0 \in V^s(\Omega), \end{aligned} \tag{2.17}$$

which is to say that $(z,r) = (y + z_e, p + r_e)$ obeys (2.14)-(2.15).
(ii) It suffices to apply the projector $P \in \mathcal{L}(\mathbf{H}^{-1}(\Omega), V_0^{-1}(\Omega))$ to (2.17), see [2, Thm. 4.8].

REMARK 3. *For $T > 0$, let us denote by $C_w([0,T], V^0(\Omega)$ the subspace in $L^\infty(0,T;V^0(\Omega))$ of functions which are continuous from $[0,T]$ into $V^0(\Omega)$ equipped with its weak topology. Then for $Py_0 \in V_n^0(\Omega)$ and for $u \in L^2(0,T;V^{3/4}(\Gamma)) \cap H^{3/4}(0,T;V^0(\Gamma))$ the existence of a solution $y \in L^2(0,T;V^1(\Omega)) \cap C_w([0,T],V^0(\Omega))$ to (2.12)-(2.13) can be obtained from [23, Thm. 5.1]. However, nothing is said there about the regularity of the pressure which appears in the Navier-Stokes equations [23, eq. (5.1) and Thm. 5.1]. The technique used in the proof of Proposition 4 to obtain a pressure term in $H^{-1/2+s/2}(\mathcal{H}^s(\Omega))$ is inspired from [24, Chap. III, Prop. 1.1, p. 266 and p. 307]. To the best of our knowledge, obtaining r in such an anisotropic Sobolev space with negative index for the time variable seems to be new. We only obtain a pressure term in $H^{-1/2+s/2}(\mathcal{H}^s(\Omega))$ because we do not have $\partial_t Py \in L^2(\mathbf{H}^{-1}(\Omega))$ but only $\partial_t Py \in L^2(V_0^{-1}(\Omega))$. This is deeply due to the fact that a Dirichlet boundary condition has to be considered because Ω is bounded [19, Chap. 3, Rem. 3.1, 4)].*

2.3. Main results. Let us state the main theorems of the paper. First, we obtain an operator $\Pi \in \mathcal{L}(V_n^0(\Omega), V_0^2(\Omega))$ solution to a Riccati equation (see also Remark 8 below).

Theorem 1 (Th. 6, Section 3). *There is a unique nonnegative and self-adjoint operator* $\Pi \in \mathcal{L}(V_n^0(\Omega))$*, which belongs to* $\mathcal{L}(V_n^0(\Omega), V_0^2(\Omega))$*, solution to the following Riccati equation:*

$$((A + A_{e,T})\Pi\xi|\zeta) + (\xi|(A + A_{e,T})\Pi\zeta) + (B^*\Pi\xi|B^*\Pi\zeta)_\Gamma = (\xi|\zeta), \tag{2.18}$$

for all $(\xi, \zeta) \in V_n^0(\Omega) \times V_n^0(\Omega)$*.*

Thus, we introduce the linear operator $(\mathcal{D}(A_\Pi), A_\Pi)$ in $V_n^0(\Omega)$ associated with Π:

$$\mathcal{D}(A_\Pi) = \big\{\xi \in V_n^0(\Omega) \mid A\xi + A_e\xi + B(B^*\Pi)\xi \in V_n^0(\Omega)\big\}, \quad A_\Pi\xi = A\xi + A_e\xi + B(B^*\Pi)\xi,$$

and we prove the following Theorem.

Theorem 2 (Th. 7, Sect. 3, Th. 9 and Cor. 6, Sect. 5). *The unbounded operator* $(\mathcal{D}(A_\Pi), A_\Pi)$ *is the infinitesimal generator of an analytic and exponentially stable semigroup on* $V_n^0(\Omega)$*, and it obeys:*

$$\mathcal{D}(A_\Pi) = \left\{ \xi \in V_n^2(\Omega) \mid \xi + PD(B^*\Pi)\xi \in V_0^2(\Omega) \right\}.$$

Moreover, we also have

$$\mathcal{D}(A_\Pi^{s/2}) = \left\{ \xi \in V_n^s(\Omega) \mid \xi + PD(B^*\Pi)\xi \in V_0^s(\Omega) \right\} \quad s \in [0, 2].$$

Next, we consider system (1.2)-(1.3) for $F = -(B^*\Pi)P$ and for an initial datum $z_0 = z_e + y_0$ where $Py_0 \in \mathcal{D}(A_\Pi^{s/2})$. According to Proposition 4, an equivalent formulation is:

$$Py' + APy + A_e Py + N(y) + B(B^*\Pi)Py = 0, \; Py(0) = Py_0 \in \mathcal{D}(A_\Pi^{s/2}), \tag{2.19}$$

$$(I - P)y = -(I - P)D(B^*\Pi)Py. \tag{2.20}$$

Notice that every function y that satisfies (2.20) can entirely be expressed in function of Py with the formula:

$$y = (I - (I - P)D(B^*\Pi))Py.$$

As a consequence, by introducing the nonlinear mapping:

$$N_\Pi : \mathcal{D}(A_\Pi^{1/2}) \longrightarrow V_0^{-1}(\Omega), \quad N_\Pi(\xi) = N\big(\xi - (I - P)D(B^*\Pi)\xi\big),$$

system (2.19)-(2.20) can be reduced to:

$$Py' + A_\Pi Py + N_\Pi(Py) = 0, \quad Py(0) = Py_0 \in \mathcal{D}(A_\Pi^{s/2}). \tag{2.21}$$

For the clarity of the exposition we rename Py by y and Py_0 by y_0 in (2.21), and we now consider the system:

$$y' + A_\Pi y + N_\Pi(y) = 0, \quad y(0) = y_0 \in \mathcal{D}(A_\Pi^{s/2}). \tag{2.22}$$

Thus, we introduce the following bilinear form which defines a new inner product on $\mathcal{D}(A_\Pi^{s/2})$:

$$(\xi|\zeta)_{\Pi,s} = \langle \Pi^{(s)}\xi|\xi\rangle_{\mathcal{D}(A_\Pi^{s/2})', \mathcal{D}(A_\Pi^{s/2})} \quad \text{where} \quad \Pi^{(s)} = A_\Pi^{*s/2+1/2}\, \Pi\, A_\Pi^{s/2+1/2}. \tag{2.23}$$

THEOREM 3 (Lem. 1, Sect. 4). *For all* $s \in [0,1]$, *the norm* $\|\cdot\|_{\Pi,s} = ((\cdot|\cdot)_{\Pi,s})^{1/2}$ *is equivalent to* $\|\cdot\|_{\mathcal{D}(A_\Pi^{s/2})}$.

The new inner product (2.23) allows to prove some useful a priori estimates for system (2.22). In particular, when $\|y_0\|_{\Pi,s}$ is small enough, the $(\cdot|\cdot)_{\Pi,s}$-product of the first equation in (2.22) with y provides an estimate which garantees that every solution to (2.22) belonging to $L^\infty_{loc}(\mathcal{D}(A_\Pi^{s/2})) \cap L^2_{loc}(\mathcal{D}(A_\Pi^{1/2+s/2}))$ decreases exponentially quickly in the norm $\|\cdot\|_{\Pi,s}$. In other words, the mapping $\xi \longmapsto \|\xi\|^2_{\Pi,s}$ is a Lyapunov function of (2.22). Hence, it allows to prove the existence and the uniqueness of the solution to (2.22), within the class of functions in $L^\infty_{loc}(\mathcal{D}(A_\Pi^{s/2})) \cap L^2_{loc}(\mathcal{D}(A_\Pi^{1/2+s/2}))$. The precise statements and results are given in the following theorem.

THEOREM 4 (Th. 11, Sect. 6). *Let* $s \in [\frac{d-2}{2}, 1]$. *There exist* $\rho_1 > 0$ *and* $\mu_1 > 0$ *such that, if* $\delta \in (0, \mu_1)$ *and*

$$y_0 \in \mathcal{I}^s_\delta = \Big\{ y \in \mathcal{D}(A_\Pi^{s/2}) \quad | \quad \|y\|_{\Pi,s} < \rho_1 \delta \Big\},$$

system (2.22) admits a solution y_{y_0} *in the set*

$$\mathcal{S}^s_\delta = \Big\{ y \in W(\mathcal{D}(A_\Pi^{1/2+s/2}), V_0^{s-1}(\Omega)) \mid \|y\|_{W(\mathcal{D}(A_\Pi^{1/2+s/2}), V_0^{s-1}(\Omega))} \leq \delta \Big\}.$$

Moreover, the solution y_{y_0} *is unique within the class of functions belonging to* $L^\infty_{loc}(\mathcal{D}(A_\Pi^{s/2})) \cap L^2_{loc}(\mathcal{D}(A_\Pi^{1/2+s/2}))$, *the mapping* $t \longmapsto \|y_{y_0}(t)\|^2_{\Pi,s}$ *decreases to* 0, *and there exists* $\sigma > 0$ *such that:*

$$\|y_{y_0}(t)\|_{\Pi,s} \leq \|y_0\|_{\Pi,s} e^{-\sigma t} \quad \forall t \geq 0. \tag{2.24}$$

Since the previous theorem can also be interpreted as a stabilization result for system (2.19)-(2.20), from Proposition 3 and Proposition 4 we deduce another theorem, in terms of partial differential equations.

THEOREM 5. *Let* Π *be the solution to (2.18), let* $f \in \mathbf{H}^1(\Omega)$ *and* $v_b \in \mathbf{H}^{5/2}(\Gamma)$ *be such that* $\int_{\Gamma_j} v_b \cdot n = 0$, *for all* $j = 1 \dots N$, *let* $(z_e, r_e) \in V^3(\Omega) \times \mathcal{H}^2(\Omega)$ *be a solution to (1.1) and let us consider the system:*

$$\partial_t z - \nu\Delta z + (z\cdot\nabla)z + \nabla r = f \ \text{ and } \ \nabla\cdot z = 0 \ \text{ in } \ Q, \quad z(0) = z_0, \tag{2.25}$$

$$z = v_b + \nu\partial_n \Pi P(z - z_e) + \psi n \ \text{ on } \ \Sigma, \tag{2.26}$$

$$\Delta\psi = \nabla\cdot((\nabla z_e)^T - z_e\cdot\nabla)\Pi P(z - z_e) \ \text{ in } \ Q, \ \int_\Gamma \psi = 0, \tag{2.27}$$

$$\partial_n\psi = (-\nu\Delta + (\nabla z_e)^T - z_e\cdot\nabla)\Pi P(z - z_e)\cdot n \ \text{ on } \ \Sigma. \tag{2.28}$$

There exist $\rho > 0$ *and* $\mu > 0$ *such that, if* $\delta \in (0, \mu)$ *and*

$$P(z_0 - z_e) \in \mathcal{W}^s_\delta = \Big\{ y \in \mathcal{D}(A_\Pi^{s/2}) \ \big| \ \|y\|_{\Pi,s} \leq \rho\delta \Big\}, \quad s \in \Big[\frac{d-2}{2}, 1\Big], \tag{2.29}$$

then (2.25)-(2.28) admits a solution (z, r) *in the set* $\{(z_e, r_e)\} + \mathcal{D}^s_\delta$ *where*

$$\mathcal{D}^s_\delta = \Big\{ (y,p) \in V^{1+s,1/2+s/2}(Q) \times H^{-1/2+s/2}(\mathcal{H}^s(\Omega)) \tag{2.30}$$

$$\|y\|_{V^{1+s,1/2+s/2}(Q)} \leq \delta, \quad \|p\|_{H^{-1/2+s/2}(\mathcal{H}^s(\Omega))} \leq \delta(1+\delta) \Big\}.$$

Moreover, the solution (z, r) *is unique within the class of functions in*

$$\big\{(z_e, r_e)\big\} + V^{1+s,1/2+s/2}_{loc}(Q) \times H^{-1/2+s/2}_{loc}(\mathcal{H}^s(\Omega)),$$

the mapping $t \mapsto \|P(z(t) - z_e)\|^2_{\Pi,s}$ *decreases to* 0*, and there is* $\sigma > 0$ *such that:*

$$\|P(z(t) - z_e)\|_{\Pi,s} \le \|P(z_0 - z_e)\|_{\Pi,s} e^{-\sigma t} \quad \forall t \ge 0. \tag{2.31}$$

Remark 4. *We can also deduce the following estimate which is stronger than (2.31), see Remark 15:*

$$\|(I - P)(z(t) - z_e)\|_{V^{1+s}(\Omega)} + \|P(z(t) - z_e)\|_{V^s_n(\Omega)} \le C\|P(z_0 - z_e)\|_{V^s_n(\Omega)} e^{-\sigma t}.$$

Remark 5. *Let us underline that in Theorem 5, it is necessary to assume* Ω *of class* C^4*, and to have a stationary state* z_e *in* $V^3(\Omega)$*. Indeed, such an assumption ensures that we have* $\mathcal{D}(A^{\theta/2}_\Pi) \hookrightarrow V^\theta_n(\Omega)$ *for all* $\theta \in [0, 2]$*, which is the main ingredient in the proof of Theorem 4, see Corollary 6 and Remark 14. The question of the smoothness of* Ω *is discussed througouht the paper, in remarks 7, 11, 12 and 14.* When $s \in [0, 1/2[$ one easily verifies that $\mathcal{D}(A^{s/2}_\Pi) = V^s_n(\Omega)$, see Remark 13. This gives the following corollary.

Corollary 1. *If* $d = 2$ *and* $s \in [0, 1/2[$*, then there exist* $\rho > 0$ *and* $\mu > 0$ *such that, if* $\delta \in (0, \mu)$ *and* $P(z_0 - z_e) \in V^s_n(\Omega)$ *obeys* $\|P(z_0 - z_e)\|_{V^s_n(\Omega)} \le \rho\delta$*, then (2.25)-(2.28) admits a solution* (z, r) *in the set* $\{(z_e, r_e)\} + \mathcal{D}^s_\delta$ *(defined by (2.30)). Moreover,* (z, r) *is unique within the class of functions in* $\{(z_e, r_e)\} + V^{1+s,1/2+s/2}_{loc}(Q) \times H^{-1/2+s/2}_{loc}(\mathcal{H}^s(\Omega))$*, the mapping* $t \mapsto \|P(z(t) - z_e)\|^2_{\Pi,s}$ *decreases to* 0*, and* z *obeys (2.31).*

Remark 6. *Corollary 1 when* $s \in [0, 1/4]$ *extends the stabilization results of [21, Thm. 6.1 and Thm. 6.7]. Notice that, in the case of tangential feedback control, an analogous result is also proved in [4, Thm. 3.1.3].*

Finally, we assume that $s \in]1/2, 1]$ and we introduce the space of initial data:

$$V^s_\Pi(\Omega) = \left\{ \xi \in V^s(\Omega) \mid \xi + D(B^*\Pi)P\xi \in V^s_0(\Omega) \right\}. \tag{2.32}$$

Notice that by recalling the characterization of B^* given in Proposition 3, the compatibility condition $\xi + D(B^*\Pi)P\xi \in V^s_0(\Omega)$ in (2.32) is equivalent to the trace condition

$$\xi = \nu\partial_n \Pi P\xi - rn \ \text{ on } \ \Gamma,$$

where r is the solution to the following Neumann problem:

$$\Delta r = \nabla\cdot(z_e\cdot\nabla - (\nabla z_e)^T)\Pi P\xi, \quad \int_\Gamma r = 0, \quad \partial_n r = (\nu\Delta - (\nabla z_e)^T + z_e\cdot\nabla)\Pi P\xi\cdot n \quad \text{on } \Gamma. \tag{2.33}$$

As a consequence, for $s \in \big]1/2, 1\big]$ the space $V^s_\Pi(\Omega)$ is also given by:

$$V^s_\Pi(\Omega) = \Big\{\xi \in V^s(\Omega) \mid \xi = \nu\partial_n \Pi P\xi - rn \ \text{ on } \ \Gamma, \ r \text{ obeys (2.33)}\Big\}. \tag{2.34}$$

From (2.32) it is easy to see that every $y_0 \in V^s_\Pi(\Omega)$ obeys $Py_0 \in \mathcal{D}(A^{s/2}_\Pi)$. It yields the following Corollary.

COROLLARY 2. *If $d = 2$ or $d = 3$ and if $s \in]1/2, 1]$, then there exist $\rho > 0$ and $\mu > 0$ such that, if $\delta \in (0, \mu)$ and $z_0 - z_e \in V^s_\Pi(\Omega)$ (given by (2.34)) obeys $\|P(z_0 - z_e)\|_{V^s_n(\Omega)} \le \rho\delta$, then (2.25)-(2.28) admits a solution (z, r) in the set $\{(z_e, r_e)\} + \mathcal{D}^s_\delta$ (defined by (2.30)). Moreover, (z, r) is unique within the class of functions in $\{(z_e, r_e)\} + V^{1+s,1/2+s/2}_{loc}(Q) \times H^{-1/2+s/2}_{loc}(\mathcal{H}^s(\Omega))$, the mapping $t \mapsto \|P(z(t) - z_e)\|^2_{\Pi,s}$ decreases to 0, and z obeys (2.31).*

3. Optimal control problem stated over an infinite time horizon. By following the path of [21], we obtain a feedback law from an auxiliary optimal control problem stated over an infinite time horizon. Let $y_0 \in V^0_n(\Omega)$ and let us consider the following minimization problem:

$$\inf\left\{\ \mathcal{J}(y,u)\ \mid\ (y,u) \in W(V^0_n(\Omega), V^{-2}_0(\Omega)) \times L^2(V^0(\Gamma))\ \text{ satisfies }\ (3.2)\right\}, \tag{3.1}$$

where

$$y' + Ay + A_e y = Bu\ \in \mathcal{D}(A + A_{e,T})', \quad y(0) = y_0 \in V^0_n(\Omega), \tag{3.2}$$

and where the cost functional $\mathcal{J}$ is defined by

$$\mathcal{J}(y,u) = \int_0^\infty \|y(t)\|^2_{V^0_n(\Omega)} dt + \int_0^\infty \|u(t)\|^2_{V^0(\Gamma)} dt. \tag{3.3}$$

THEOREM 6. *For all $y_0 \in V^0_n(\Omega)$, problem (3.1) admits a unique solution (y_{y_0}, u_{y_0}). The optimal control obeys $u_{y_0} = -B^*\Phi_{y_0}$, where the pair $(y_{y_0}, \Phi_{y_0}) \in W(V^0_n(\Omega), V^{-2}_0(\Omega)) \times W(V^2_0(\Omega), V^0_n(\Omega))$ is the unique solution to:*

$$(\mathcal{S}_{y_0}) \left\{ \begin{array}{lll} y' + Ay + A_e y & = -BB^*\Phi, & y(0) = y_0 \in V^0_n(\Omega), \\ -\Phi' + A\Phi + A_{e,T}\Phi & = y, & \Phi(\infty) = 0, \\ \Phi(t) & = \Pi\, y(t) & \forall t \ge 0. \end{array} \right.$$

In the above setting, Π is the unique nonnegative and self-adjoint operator of $\mathcal{L}(V^0_n(\Omega))$, which belongs to $\mathcal{L}(V^0_n(\Omega), V^2_0(\Omega))$, solution to the following Riccati equation:

$$((A + A_{e,T})\Pi\xi|\zeta) + (\xi|(A + A_{e,T})\Pi\zeta) + (B^*\Pi\xi|B^*\Pi\zeta)_\Gamma = (\xi|\zeta), \tag{3.4}$$

for all $(\xi, \zeta) \in V^0_n(\Omega) \times V^0_n(\Omega)$. Moreover, Π obeys:

$$(\Pi y_0|y_0) = \mathcal{J}(y_{y_0}, u_{y_0}) = \inf\left\{\ \mathcal{J}(y,u)\ \mid\ (y,u)\ \textit{ satisfies }\ (3.2)\right\}.$$

Proof. This theorem can be deduced from an obvious adaptation of Theorem 4.1, of Lemma 4.2 and of Theorem 4.5 in [21] (there R_A and M shall be replace by the identity in $\mathbf{V}^0(\Gamma)$).

REMARK 7. *The assumptions Ω of class C^4 and $z_e \in V^3(\Omega)$ are required in [21] to study the regularity of the solution to $(\mathcal{S}_{y_0})$, in order to obtain the boundedness of Π from $V^0_n(\Omega)$ into $V^2_0(\Omega)$, see [21, Rem. 4.4]. However, according to [5, App. B.4] it is possible to obtain the boundedness of Π from $V^0_n(\Omega)$ into $\mathcal{D}(A)$ without assuming that Ω is smooth. Hence, Theorem 6 remains valid if Ω is only of class C^2 or if Ω is a convex polyhedral domain, because for such geometrical assumptions we still have $\mathcal{D}(A) = V^2_0(\Omega)$ [10].*

Remark 8. *The theory of [18] only guarantees that* Π *is bounded from* $V^0_n(\Omega)$ *into* $\mathcal{D}(A^{1-\epsilon})$ *for* $\epsilon > 0$*, and that Riccati equation (3.4) is well-posed for* $(\xi,\zeta) \in \mathcal{D}(A^\epsilon) \times \mathcal{D}(A^\epsilon)$ *[18, Chap.2, Thm 2.2.1 and Thm. 2.3.9.1 step 2]. However, in the later work [5, App. Prop. B.4.1] it is proved that* Π *is also bounded from* $V^0_n(\Omega)$ *into* $\mathcal{D}(A)$*. As a consequence, we have* $(A + A_{e,T})\Pi \in \mathcal{L}(V^0_n(\Omega))$ *and* $B^*\Pi \in \mathcal{L}(V^0(\Gamma), V^0_n(\Omega))$*, and (3.4) is well-defined for* $(\xi,\zeta) \in V^0_n(\Omega) \times V^0_n(\Omega)$*. In fact, by using the self-adjointness of* Π *one can also prove that it can be extended to a bounded operator from* $\mathcal{D}(A)'$ *into* $V^0_n(\Omega)$*, and that* $\Pi(A + A_e) \in \mathcal{L}(V^0_n(\Omega))$ *and* $\Pi B \in \mathcal{L}(V^0_n(\Omega), V^0(\Gamma))$*. Hence, (3.4) can also be interpreted as the following equation stated in* $\mathcal{L}(V^0_n(\Omega))$*:*

$$(A + A_{e,T})\Pi + \Pi(A + A_e) + \Pi BB^*\Pi = I,$$

where I *denotes the identity in* $V^0_n(\Omega)$*. See also [5, App. Prop. B.4.1, (B.4.11)] where the Riccati equation is written with* $(B^*\Pi)^*B^*\Pi$ *instead of* $\Pi BB^*\Pi$*.*

Since Π belongs to $\mathcal{L}(V^0_n(\Omega), V^2_0(\Omega))$, the operator $B^*\Pi$ belongs to $\mathcal{L}(V^0_n(\Omega), V^0(\Gamma))$ and $B(B^*\Pi)$ is well-defined as a bounded operator from $V^0_n(\Omega)$ into $V^{-2}_0(\Omega)$. As a consequence, from $y_{y_0} \in W(V^0_n(\Omega), V^{-2}_0(\Omega))$ we deduce that $B(B^*\Pi)y_{y_0} \in L^2(V^{-2}_0(\Omega))$ and we are allowed to replace Φ_{y_0} by Πy_{y_0} in the first equality of $(\mathcal{S}_{y_0})$. It means that the optimal state y_{y_0} obeys:

$$\frac{d}{dt}(y_{y_0}(t)|v) + (y_{y_0}(t)|A\zeta + A_{e,T}\zeta + (B^*\Pi)^*B^*\zeta) = 0,\ y_{y_0}(0) = y_0 \in V^0_n(\Omega), \quad (3.5)$$

for all $\zeta \in V^2_0(\Omega)$ and $t \geq 0$.

Moreover, with a bootstrap argument based on system $(\mathcal{S}_{y_0})$, one can prove that y_{y_0} is time continuous in $V^0_n(\Omega)$ [21, Lem. 4.2, step 4 of the proof]. Hence, since the solution to $(\mathcal{S}_{y_0})$ is unique, it is easy to see that the family $(S(t))_{t\geq 0}$ defined by $S(t) : y_0 \longmapsto y_{y_0}(t)$ is a strongly continuous semigroup on $V^0_n(\Omega)$, see also [18, Chap. 2, Th. 2.4.6.1]. So we may prefer the notation $(S(t))_{t\geq 0} = (e^{-A_\Pi t})_{t\geq 0}$, where $(\mathcal{D}(A_\Pi), -A_\Pi)$ is the infinitesimal generator of $(S(t))_{t\geq 0}$. It means that the optimal state y_{y_0} also obeys:

$$y_{y_0}(t) = e^{-A_\Pi t}y_0, \quad \forall t \geq 0. \quad (3.6)$$

However, in order that formula (3.6) be more explicit, we shall give a characterization of $(\mathcal{D}(A_\Pi), A_\Pi)$ in terms of the linear operators appearing in equation (3.5). For all $\xi \in V^0_n(\Omega)$ let us define $A\xi + A_e\xi + B(B^*\Pi)\xi$ as the following element of $V^{-2}_0(\Omega)$:

$$\langle A\xi + A_e\xi + B(B^*\Pi)\xi|\zeta\rangle_{V^{-2}_0(\Omega),V^2_0(\Omega)} = (\xi|A\zeta + A_{e,T}\zeta + (B^*\Pi)^*B^*\zeta), \quad (3.7)$$

for all $\zeta \in V^2_0(\Omega)$. Hence, the following equalities can be proved:

$$\mathcal{D}(A_\Pi) = \big\{\xi \in V^0_n(\Omega) \mid A\xi + A_e\xi + B(B^*\Pi)\xi \in V^0_n(\Omega)\big\}, \quad (3.8)$$

$$A_\Pi\xi = A\xi + A_e\xi + B(B^*\Pi)\xi. \quad (3.9)$$

The above characterization is obtained by combining (3.5) and (3.6). First, for all $\xi \in \mathcal{D}(A_\Pi)$, by setting $y_0 = \xi$ and $y(t) = e^{-A_\Pi t}\xi$ in (3.5) we obtain:

$$(e^{-A_\Pi t}\xi|A\zeta + A_{e,T}\zeta + (B^*\Pi)^*B^*\zeta) = -\frac{d}{dt}(e^{-A_\Pi t}\xi|\zeta) = (A_\Pi e^{-A_\Pi t}\xi|\zeta),$$

for all $(\xi,\zeta) \in \mathcal{D}(A_\Pi) \times V^2_0(\Omega)$. Thus, by setting $t = 0$ in the above equation we deduce the following equality:

$$(\xi|A\zeta + A_{e,T}\zeta + (B^*\Pi)^*B^*\zeta) = (A_\Pi\xi|\zeta) \quad \forall(\xi,\zeta) \in \mathcal{D}(A_\Pi) \times V^2_0(\Omega), \quad (3.10)$$

which implies $V_0^2(\Omega) \subset \mathcal{D}(A_\Pi^*)$ and $A_\Pi^*\zeta = (A + A_{e,T} + (B^*\Pi)^*B^*)\zeta$ for all $\zeta \in V_0^2(\Omega)$. The converse inclusion $\mathcal{D}(A_\Pi^*) \subset V_0^2(\Omega)$ can be obtained as follows. Assume that ζ belongs to $\mathcal{D}(A_\Pi^*)$ and, for $\lambda > 0$ in the resolvent sets of $(\mathcal{D}(A_\Pi^*), A_\Pi^*)$ and of $(V_0^2(\Omega), A + A_{e,T} + (B^*\Pi)^*B^*)$, consider the solution $\widetilde{\zeta} \in V_0^2(\Omega)$ to:

$$(\lambda + A + A_{e,T} + (B^*\Pi)^*B^*)\widetilde{\zeta} = (\lambda + A_\Pi^*)\zeta \in V_n^0(\Omega). \tag{3.11}$$

Notice that such a $\lambda > 0$ exists because $(\mathcal{D}(A_\Pi^*), A_\Pi^*)$ and $(V_0^2(\Omega), A+A_{e,T}+(B^*\Pi)^*B^*)$ are both infinitesimal generators of strongly continuous semigroups on $V_n^0(\Omega)$ [20, Cor.3.8]. Indeed, the existence and the strong continuity of $(e^{-A_\Pi^* t})_{t\geq 0}$ follow with a duality argument [20, Cor.10.6], and since $(B^*\Pi)^*B^*$ belongs to $\mathcal{L}(\mathcal{D}(A^{3/4+\varepsilon}), V_n^0(\Omega))$ for $0 < \varepsilon < 1/4$, a perturbation argument ensures that $(V_0^2(\Omega), A + A_{e,T} + (B^*\Pi)^*B^*)$ is the infinitesimal generator of an analytic semigroup on $V_n^0(\Omega)$ [20, Chap.3, Cor.2.4]. Hence, by combining (3.10) and (3.11) we deduce that $(\lambda + A_\Pi^*)\widetilde{\zeta} = (\lambda + A_\Pi^*)\zeta$, which yields $\zeta = \widetilde{\zeta} \in V_0^2(\Omega)$, and the converse inclusion $\mathcal{D}(A_\Pi^*) \subset V_0^2(\Omega)$ is proved. As a consequence, we have:

$$\mathcal{D}(A_\Pi^*) = V_0^2(\Omega) \quad \text{and} \quad A_\Pi^* = A + A_{e,T} + (B^*\Pi)^*B^*, \tag{3.12}$$

which, in view of (3.7), exactly means (3.8)-(3.9).

Finally, we shall also recall that $(e^{-A_\Pi t})_{t\geq 0}$ is exponentially stable on $V_n^0(\Omega)$. Indeed, since Theorem 6 states that $y_{y_0} = e^{-A_\Pi(\cdot)}y_0$ belongs to $L^2(V_n^0(\Omega))$ for all $y_0 \in V_n^0(\Omega)$, the exponential stability of $(e^{-A_\Pi t})_{t\geq 0}$ follows from a well known result due to Datko [20, Chap. 4, Th. 4.1].

Let us summerize those results in the following theorem.

THEOREM 7. *The linear operator $(\mathcal{D}(A_\Pi), A_\Pi))$ defined by (3.8)-(3.9) is the infinitesimal generator of an analytic and exponentially stable semigroup on $V_n^0(\Omega)$, and the adjoint of $(\mathcal{D}(A_\Pi), A_\Pi))$ is given by (3.12). Moreover, for $y_0 \in V_n^0(\Omega)$ the optimal trajectory y_{y_0} satisfies $y_{y_0}(t) = e^{-A_\Pi t}y_0$ for all $t \geq 0$. It means that y_{y_0} is the unique solution to:*

$$y' + A_\Pi y = 0, \quad y(0) = y_0 \in V_n^0(\Omega),$$

and that there exists $\sigma > 0$ such that:

$$\|y_{y_0}(t)\|_{V_n^0(\Omega)} \leq Ce^{-\sigma t}\|y_0\|_{V_n^0(\Omega)} \quad \forall t \geq 0.$$

4. A Lyapunov function for the closed-loop Oseen system. Let us assume that $s \in [0, 1]$ and let us consider the closed-loop Oseen system:

$$y' + A_\Pi y = 0, \quad y(0) = y_0 \in \mathcal{D}(A_\Pi^{s/2}), \tag{4.1}$$

for which, according to Theorem 7, the solution is known to be exponentially stable. The main goal of the present section is to exhibit a Lyapunov function for system (4.1). More precisely, we want to find a function $\mathcal{V}_s(\cdot)$, defined on $\mathcal{D}(A_\Pi^{s/2})$ and with values in $\mathbb{R}^+$, which satisfies:

$$\mathcal{V}_s(\xi) \simeq \|\xi\|^2_{\mathcal{D}(A_\Pi^{s/2})} \quad \text{and} \quad t \longmapsto \mathcal{V}_s(e^{-A_\Pi t}\xi) \text{ decreases to } 0. \tag{4.2}$$

In a first step, we need to prove the following proposition.

Proposition 5. *For all $s \in [0,2]$, the following equalities hold:*

$$\mathcal{D}(A_\Pi^{s/2}) = [\mathcal{D}(A_\Pi), V_n^0(\Omega)]_{1-s/2} \quad \text{and} \quad \mathcal{D}(A_\Pi^{*s/2}) = [\mathcal{D}(A_\Pi^*), V_n^0(\Omega)]_{1-s/2}. \tag{4.3}$$

Proof. According to [26], (4.3) is true, if and only if, the holomorphic function

$$z \in \{z \in \mathbb{C} \mid Re(z) > 0\} \longmapsto {A_\Pi}^{-z} \in \mathcal{L}(V_n^0(\Omega)),$$

can be extended to a strong continuous function from $\{z \in \mathbb{C} \mid Re(z) \geq 0\}$ in $\mathcal{L}(V_n^0(\Omega))$. First, because $(e^{-A_\Pi t})_{t\geq 0}$ is analytic an exponentially stable on $V_n^0(\Omega)$, there exists $0 < \theta < \frac{\pi}{2}$ such that $\{\lambda \in \mathbb{C},\ \theta \leq |arg(\lambda)| \leq \pi\} \subset \rho(A_\Pi)$, and for $z \in \mathbb{C}$ obeying $Re(z) > 0$ the operator A_Π^{-z} is given by the Cauchy's integral formula:

$$A_\Pi^{-z} = \frac{1}{2i\pi} \int_{\mathcal{C}_\theta} \lambda^{-z}(\lambda - A_\Pi)^{-1} d\lambda,$$

where $\mathcal{C}_\theta$ is an oriented path which runs in the resolvent set of A_Π from $\infty e^{i\theta}$ to $\infty e^{-i\theta}$, avoiding the negative real axis and the origin. Since the integral converges in the uniform operator topology when $Re(z) > 0$, it defines a bounded operator A_Π^{-z}. About the above formula, one may refer to [20, Chap 2, 2.6] when $z = \alpha$ is a positive real value, to the extended operational calculus theory of [17, Sec. 5.11] or to the extended functional calculus recalled in [11]. Thus, for $0 < Re(z) < 1$ we can deform the path of integration C_θ to the lower and upper sides of the negative real axis and obtain:

$$A_\Pi^{-z} = \frac{\sin \pi z}{\pi} \int_0^{+\infty} t^{-z}(t + A_\Pi)^{-1} dt \quad \text{for} \quad 0 < Re(z) < 1.$$

Next, from the perturbation equality:

$$(t + A_\Pi)^{-1} = (t + A)^{-1} - (t + A)^{-1}(A_e + B(B^*\Pi))(t + A_\Pi)^{-1} \quad \forall t \geq 0,$$

we deduce that

$$A_\Pi^{-z} = A^{-z} - I(z) \ \text{ where } \ I(z) = \frac{\sin \pi z}{\pi} \int_0^{+\infty} t^{-z}(t+A)^{-1}(A_e + B(B^*\Pi))(t+A_\Pi)^{-1} dt.$$

Moreover, because A is self-adjoint, the interpolation equality $\mathcal{D}(A^{s/2}) = [\mathcal{D}(A), V_n^0(\Omega)]_{1-s/2}$ is true for all $s \in [0,2]$ [8, Chap. 1, Prop. 6.1] and from [26] we deduce that A^{-z} can be extended to a strong continuous function from $\{z \in \mathbb{C} \mid Re(z) \geq 0\}$ in $\mathcal{L}(V_n^0(\Omega))$. As a consequence, A^{-z} is bounded independently on $z \in \{z \in \mathbb{C} \mid Re(z) > 0\}$ in a neighborhood of 0, and by virtue of [17, Ch. 17, Thm. 17.9.1], it remains to show that $z \longmapsto I(z)$ is bounded independently on $z \in \{z \in \mathbb{C} \mid Re(z) > 0\}$ in a neighborhood of 0. Let ρ and σ be respectively the real and the imaginary part of $z \in \{z \in \mathbb{C} \mid Re(z) \geq 0\}$. Since A and A_Π are infinitesimal generators of analytic semigroups on $V_n^0(\Omega)$, for $\varepsilon \in]0, 1/4[$ we have the following resolvent inequalities:

$$\|(t + A_\Pi)^{-1}\|_{V_n^0(\Omega)} \leq \frac{C}{1+t} \quad \text{and} \quad \|(t + A)^{-1} A^{3/4+\varepsilon}\|_{V_n^0(\Omega)} \leq \frac{C}{(1+t)^{1/4-\varepsilon}} \quad \forall t \geq 0,$$

and with $A^{-3/4-\varepsilon}(A_e+B(B^*\Pi))\in\mathcal{L}(V^0_n(\Omega))$, we conclude that $\|I(z)\|_{\mathcal{L}(V^0_n(\Omega))}$, which is explicitly given by:

$$\left\|\frac{\sin\pi z}{\pi}\int_0^{+\infty} t^{-z}\big((t+A)^{-1}A^{3/4+\varepsilon}\big)\big(A^{-3/4-\varepsilon}(A_e+B(B^*\Pi))\big)(t+A_\Pi)^{-1}dt\right\|_{\mathcal{L}(V^0_n(\Omega))}$$

is bounded by $Ce^{\pi\sigma}\displaystyle\int_0^{\infty}\frac{dt}{t^{-\rho}(1+t)^{5/4-\varepsilon}}$.
Proposition 5 has several important consequences. The first obvious one is stated in the following corollary.

Corollary 3. *The unbounded operator* $(\mathcal{D}(A^*_\Pi),A^*_\Pi)$ *obeys:*

$$\mathcal{D}(A^{*s/2}_\Pi)=V^s_0(\Omega)\quad\forall s\in[0,2]. \tag{4.4}$$

Proof. Equality (4.4) follows from the second equality in (4.3) with $\mathcal{D}(A^*_\Pi)=V^2_0(\Omega)$.

Remark 9. *Notice that for* $s\in[0,2]$*, the space* $\mathcal{D}(A^{*s/2}_\Pi)$ *is the range of the operator* $A^{*-s/2}_\Pi\in\mathcal{L}(V^0_n(\Omega))$ *[20, Chap. 2, Thm. 6.8]. As a consequence,* $A^{*-s/2}_\Pi$ *is an isomorphism from* $V^0_n(\Omega)$ *onto* $\mathcal{D}(A^{*s/2}_\Pi)$ *and we can extend the definition of* $A^{-s/2}_\Pi$ *to a bounded linear operator from* $\mathcal{D}(A^{*s/2}_\Pi)'$ *onto* $V^0_n(\Omega)$ *as follows:*

$$(A^{-s/2}_\Pi\xi|\zeta)=\langle\xi|A^{*-s/2}_\Pi\zeta\rangle_{\mathcal{D}(A^{*s/2}_\Pi)',\mathcal{D}(A^{*s/2}_\Pi)}\qquad\text{for all }\ (\xi,\zeta)\in\mathcal{D}(A^{*s/2}_\Pi)'\times V^0_n(\Omega).$$

Moreover, we easily verify that $\|A^{-s/2}_\Pi\cdot\|_{V^0_n(\Omega)}$ *defines a norm which is equivalent to the one of* $\mathcal{D}(A^{*s/2}_\Pi)'$*:*

$$\|A^{-s/2}_\Pi\cdot\|_{V^0_n(\Omega)}\sim\|\cdot\|_{\mathcal{D}(A^{*s/2}_\Pi)'}\sim\|\cdot\|_{V^{-s}_0(\Omega)}\qquad\text{for all } s\in[0,2]. \tag{4.5}$$

The second important consequence of Proposition 5 is the following regularity result for the nonhomogeneous equation which is needed in Section **6** to study the nonlinear closed-loop system.

Corollary 4. *Let* $s\in[0,1]$*,* $y_0\in\mathcal{D}(A^{s/2}_\Pi)$ *and* $f\in L^2(V^{s-1}_0(\Omega))$*. The solution to*

$$y'+A_\Pi y=f\qquad\text{and}\qquad y(0)=y_0,$$

belongs to $W(\mathcal{D}(A^{1/2+s/2}_\Pi),V^{s-1}_0(\Omega))$ *and obeys the following estimate:*

$$\|y\|_{W(\mathcal{D}(A^{1/2+s/2}_\Pi),V^{s-1}_0(\Omega))}\le C(\|f\|_{L^2(V^{s-1}_0(\Omega))}+\|y_0\|_{\mathcal{D}(A^{s/2}_\Pi)}). \tag{4.6}$$

Proof. Maximal regularity results for analytic semigroup [8, Chap. 3, Thm. 2.2] ensure that the mapping:

$$\begin{array}{rcl} W([\mathcal{D}(A_\Pi),V^0_n(\Omega)]_{1/2-s/2},[\mathcal{D}(A^*_\Pi),V^0_n(\Omega)]'_{s/2+1/2}) & \to & L^2([\mathcal{D}(A^*_\Pi),V^0_n(\Omega)]'_{s/2+1/2}) \\ & & \times I_{s,1/2} \\ y & \mapsto & (y'+A_\Pi y\,,\,y(0)), \end{array}$$

is an isomorphism. In the above setting $I_{s,1/2} = \big[[\mathcal{D}(A_\Pi), V^0_n(\Omega)]_{1/2-s/2}, [\mathcal{D}(A^*_\Pi), V^0_n(\Omega)]'_{s/2+1/2}\big]_{1/2}$. Notice that we can set $T = +\infty$ in [8, Chap. 3, Thm. 2.2] because $-A_\Pi$ is of negative type. Hence, since from (4.3) and (4.4) we deduce that

$$[\mathcal{D}(A_\Pi), V^0_n(\Omega)]_{1/2-s/2} = \mathcal{D}(A_\Pi^{s/2+1/2}) \quad \text{and} \quad [\mathcal{D}(A^*_\Pi), V^0_n(\Omega)]'_{s/2+1/2} = V^{s-1}_0(\Omega),$$

it remains to prove that $I_{s,1/2} = \mathcal{D}(A_\Pi^{s/2})$ to obtain the desired result. First, from (4.3) we have:

$$I_{s,1/2} = [\mathcal{D}(A_\Pi^{1/2+s/2}), \mathcal{D}(A_\Pi^{*1/2-s/2})']_{1/2},$$

and since $A_\Pi^{1/2-s/2}$ is an isomorphism from $\mathcal{D}(A_\Pi)$ onto $\mathcal{D}(A_\Pi^{1/2+s/2})$ as well as from $V^0_n(\Omega)$ onto $\mathcal{D}(A_\Pi^{*1/2-s/2})'$, an interpolation argument yields:

$$I_{s,1/2} = [\mathcal{D}(A_\Pi^{1/2+s/2}), \mathcal{D}(A_\Pi^{*1/2-s/2})']_{1/2} = A_\Pi^{1/2-s/2}\big([\mathcal{D}(A_\Pi), V^0_n(\Omega)]_{1/2}\big).$$

Finally, we conclude with the first equality in (4.3) which allows to make the following calculation:

$$A_\Pi^{1/2-s/2}\big([\mathcal{D}(A_\Pi), V^0_n(\Omega)]_{1/2}\big) = A_\Pi^{1/2-s/2}\big(\mathcal{D}(A_\Pi^{1/2})\big) = \mathcal{D}(A_\Pi^{s/2}).$$

The third consequence of Proposition 5 is the following regularity property for the linear operator $A_\Pi^{*s/2+1/2}\,\Pi\,A_\Pi^{s/2+1/2}$, which is the main tool to construct a function $\mathcal{V}_s(\cdot)$ satisfying (4.2).

Corollary 5. *The linear operator* Π *obeys:*

$$A_\Pi^{*s/2+1/2}\,\Pi\,A_\Pi^{s/2+1/2} \in \mathcal{L}(\mathcal{D}(A_\Pi^{s/2}), \mathcal{D}(A_\Pi^{s/2})') \quad \forall s \in [0,1]. \tag{4.7}$$

Proof. Because $\Pi \in \mathcal{L}(V^0_n(\Omega))$ is a self-adjoint operator which belongs to $\mathcal{L}(V^0_n(\Omega), V^2_0(\Omega))$, a duality argument yields $\Pi \in \mathcal{L}(V^{-2}_0(\Omega), V^0_n(\Omega))$, and $\Pi \in \mathcal{L}(V^{-1}_0(\Omega), V^1_0(\Omega))$ follows by interpolation. Hence, from (4.4) with $s = 1$ we deduce that $A_\Pi^{*1/2} \in \mathcal{L}(\mathcal{D}(A_\Pi^{*1/2}), V^0_n(\Omega)) = \mathcal{L}(V^1_0(\Omega), V^0_n(\Omega))$ and $A_\Pi^{1/2} \in \mathcal{L}(V^0_n(\Omega), \mathcal{D}(A_\Pi^{*1/2})')$ $= \mathcal{L}(V^0_n(\Omega), V^{-1}_0(\Omega))$. It follows that $A_\Pi^{*1/2}\,\Pi\,A_\Pi^{1/2} \in \mathcal{L}(V^0_n(\Omega))$ from which (4.7) is a direct consequence.

According to Corollary 5, the following definition is consistent.

Definition 2. *For* $s \in [0,1]$, *let us define* $\Pi^{(s)} \in \mathcal{L}(\mathcal{D}(A_\Pi^{s/2}), \mathcal{D}(A_\Pi^{s/2})')$ *by* $\Pi^{(s)} = A_\Pi^{*s/2+1/2}\,\Pi\,A_\Pi^{s/2+1/2}$.

Remark 10. *As in the proof of Corollary 5, an interpolation argument gives* $\Pi \in \mathcal{L}(V^{s-1}_0(\Omega), V^{s+1}_0(\Omega))$ *which guarantees that* $A_\Pi^{*s/2+1/2}\,\Pi\,A_\Pi^{s/2+1/2}$ *belongs to* $\mathcal{L}(\mathcal{D}(A_\Pi^{s}), V^0_n(\Omega))$. *Hence, because from* $\Pi \in \mathcal{L}(V^0_n(\Omega), V^2_0(\Omega))$ *we also deduce that* $A_\Pi^{*s/2+1/2}\,\Pi\,A_\Pi^{s/2+1/2} \in \mathcal{L}(\mathcal{D}(A_\Pi^{1/2+s/2}), V^{1-s}_0(\Omega))$, *an interpolation argument yields:*

$$\Pi^{(s)} \in \mathcal{L}(\mathcal{D}(A_\Pi^{\theta/2+s/2}), V^{\theta-s}_0(\Omega)) \quad 0 \le s \le \theta \le 1. \tag{4.8}$$

The linear operator $\Pi^{(s)}$ allows to construct a function $\mathcal{V}_s(\cdot)$ which satisfies (4.2). Indeed, we can prove that the bilinear form $(\cdot|\cdot)_{\Pi,s}$ defined by (4.9) below is a scalar product on $\mathcal{D}(A_\Pi^{s/2})$ for which A_Π is accretive:

$$(\xi|\zeta)_{\Pi,s} = \langle \Pi^{(s)}\xi|\zeta\rangle_{(\mathcal{D}(A_\Pi^{s/2}))',\mathcal{D}(A_\Pi^{s/2})} \quad \text{for all} \quad (\xi,\zeta) \in \mathcal{D}(A_\Pi^{s/2}) \times \mathcal{D}(A_\Pi^{s/2}). \tag{4.9}$$

More precisely, from (3.4) we deduce an expression of $(A_\Pi\xi|\xi)_{\Pi,s}$ which allows to prove that

$$(A_\Pi\xi|\xi)_{\Pi,s} \geq \sigma(\xi|\xi)_{\Pi,s},$$

for some $\sigma > 0$. Hence, from the calculation of $(y' + A_\Pi y|y)_{\Pi,s}$ we obtain that the mapping

$$\xi \longmapsto \mathcal{V}_s(\xi) = (\xi|\xi)_{\Pi,s}$$

satisfies (4.2) and that $t \mapsto \mathcal{V}_s(e^{-A_\Pi t}\xi)$ has an exponential rate of decrease equal to $2\sigma > 0$.

LEMMA 1. *For all $s \in [0,1]$, the bilinear form $(\cdot|\cdot)_{\Pi,s}$ defined by (4.9) is a scalar product on $\mathcal{D}(A_\Pi^{s/2})$. If we define $\|\xi\|_{\Pi,s} = \big((\xi|\xi)_{\Pi,s}\big)^{1/2}$, then the following norm equivalence holds:*

$$\|\cdot\|_{\Pi,s} \sim \|\cdot\|_{\mathcal{D}(A_\Pi^{s/2})}. \tag{4.10}$$

Moreover, we also have:

$$\big((A_\Pi\cdot|\cdot)_{\Pi,s}\big)^{1/2} \sim \|\cdot\|_{\mathcal{D}(A_\Pi^{1/2+s/2})}. \tag{4.11}$$

Proof. *(i) Norm equivalence (4.10).*
Because for all $\xi \in \mathcal{D}(A_\Pi^{s/2})$ we have the equality $\|\xi\|_{\Pi,s} = \|A_\Pi^{s/2}\xi\|_{\Pi,0}$, it is sufficient to prove (4.10) for $s = 0$. First, $\|\cdot\|_{\Pi,0} \leq C\|\cdot\|_{V_n^0(\Omega)}$ is a straightforward consequence of $\Pi^{(0)} \in \mathcal{L}(V_n^0(\Omega))$. To prove the converse inequality, let us first pick $\xi \in V_n^0(\Omega)$ and set $\zeta = A_\Pi^{1/2}\xi \in \mathcal{D}(A_\Pi^{-1/2})$. Hence, from (4.5) for $s = 1$ and from the equalities $\mathcal{D}(A_\Pi^{*1/2})' = [V_n^0(\Omega), \mathcal{D}(A_\Pi^*)]'_{1/2} = [V_n^0(\Omega), \mathcal{D}(A_\Pi^*)']_{1/2}$ we have

$$\|\xi\|_{V_n^0(\Omega)} = \|A_\Pi^{-1/2}\zeta\|_{V_n^0(\Omega)} \leq C\|\zeta\|_{[V_n^0(\Omega),\mathcal{D}(A_\Pi^*)']_{1/2}}.$$

As a consequence, since $[V_n^0(\Omega), \mathcal{D}(A_\Pi^*)']_{1/2}$ is the trace space of $W(V_n^0(\Omega), \mathcal{D}(A_\Pi^*)')$ we deduce that:

$$\begin{aligned}\|\xi\|_{V_n^0(\Omega)} &\leq C\|e^{-A_\Pi t}\zeta\|_{W(V_n^0(\Omega),\mathcal{D}(A_\Pi^*)')} \\ &= C(\|e^{-A_\Pi t}\zeta\|_{L^2(V_n^0(\Omega))} + \|\frac{d}{dt}e^{-A_\Pi t}\zeta\|_{L^2(\mathcal{D}(A_\Pi^*)')}).\end{aligned}$$

Thus, (4.5) for $s = 2$ with $\frac{d}{dt}e^{-A_\Pi t}\zeta = -A_\Pi e^{-A_\Pi t}\zeta$ yield

$$\|\xi\|_{V_n^0(\Omega)} \leq C\|e^{-A_\Pi t}\zeta\|_{L^2(V_n^0(\Omega))},$$

and we conclude by observing that:

$$\|e^{-A_\Pi t}\zeta\|^2_{L^2(V_n^0(\Omega))} + \|(B^*\Pi)e^{-A_\Pi t}\zeta\|^2_{L^2(V_n^0(\Omega))} = (\Pi\zeta|\zeta) = (\Pi^{(0)}\xi|\xi) = \|\xi\|^2_{\Pi,0}.$$

(ii) Norm equivalence (4.11).
If we replace ξ and ζ by $A_\Pi^{s/2+1/2}\xi$ in (3.4), we obtain:

$$\begin{aligned}(A_\Pi\xi|\xi)_{\Pi,s} &= \langle A_\Pi\xi|\Pi^{(s)}\xi\rangle_{\mathcal{D}(A_\Pi^{1/2-s/2})',\mathcal{D}(A_\Pi^{*1/2-s/2})}\\ &= \frac{1}{2}\|A_\Pi^{s/2+1/2}\xi\|^2_{V_n^0(\Omega)} + \frac{1}{2}\|(B^*\Pi)A_\Pi^{s/2+1/2}\xi\|^2_{V^0(\Gamma)},\end{aligned}$$

for all $\xi \in \mathcal{D}(A_\Pi^{1/2+s/2})$. Hence, (4.11) follows because $B^*\Pi \in \mathcal{L}(V_n^0(\Omega), V^0(\Gamma))$.
Finally, we are in position to state the main theorem of the present section.

THEOREM 8. *For all $s \in [0,1]$, the mapping $\mathcal{V}_s(\cdot)$ defined on $\mathcal{D}(A_\Pi^{s/2})$ by $\mathcal{V}_s(\xi) = \|\xi\|^2_{\Pi,s}$ satisfies (4.2).*

Proof. The continuous embedding $\mathcal{D}(A_\Pi^{1/2+s/2}) \hookrightarrow \mathcal{D}(A_\Pi^{s/2})$ with (4.10) and (4.11) provides $\sigma > 0$ such that $(A_\Pi\xi|\xi)_{\Pi,s} \geq \sigma\|\xi\|^2_{\Pi,s}$. Hence, we conclude by remarking that $(y' + A_\Pi y|y)_{\Pi,s} = 0$ implies:

$$\frac{d}{dt}\|y(t)\|^2_{\Pi,s} + 2\sigma\|y(t)\|^2_{\Pi,s} \leq 0.$$

5. A class of initial data. The main goal of this section is to give a characterization of $\mathcal{D}(A_\Pi^{s/2})$ for $s \in [0,2]$. Corollary 6 below states that $\mathcal{D}(A_\Pi^{s/2})$ is composed of elements ξ of $V_n^s(\Omega)$ which satisfy the compatibility condition $\xi + PD(B^*\Pi)\xi \in V_0^s(\Omega)$. We underline that the results of this section rely on the geometrical assumption Ω of class C^4.

Notice that the analysis of [21] does not give a characterization of $\mathcal{D}(A_\Pi)$ better than (3.8)-(3.9), and the approach in [18, Chap. 2] is too general and only allows to characterize $\mathcal{D}(A_\Pi)$ in terms of the domain of the free dynamic operator $A + A_e$. The main information on $\mathcal{D}(A_\Pi)$ obtained there is the inclusion:

$$\mathcal{D}(A_\Pi) \subset \mathcal{D}((\lambda_0 + A + A_e)^{1/4-\varepsilon}) \quad \text{for} \quad \varepsilon \in \left]0, 1/4\right[,$$

which is a consequence of [18, Chap. 2, Thm. 2.2.1, (a_6)]. Even when they consider the heat equation with a Dirichlet boundary control in [18, Chap. 3, Par. 3.1 and 3.2], the authors do not provide a complete characterization of the domain of the closed-loop operator. The only statement dealing with a characterization of a Riccati-based closed-loop operator is [4, Prop. 6.1], but the question of the regularity of Γ is avoided, see Remark 12 *(ii)* below.

Let us first prove two preliminary lemmas.

LEMMA 2. *The operator Π is a linear continuous operator from*

$$\left\{ \xi \in V_n^1(\Omega) \mid \xi + PD(B^*\Pi)\xi \in V_0^1(\Omega) \right\} \tag{5.1}$$

onto $V_0^3(\Omega)$.

Proof. We assume that ξ belongs to the space defined by (5.1) and we define $\widetilde{y} = e^{-\lambda_0(\cdot)}y_\xi$ and $\widetilde{\Phi} = e^{-\lambda_0(\cdot)}\Phi_\xi$, where (y_ξ, Φ_ξ) is the unique solution to $(\mathcal{S}_{y_0})$ for $y_0 = \xi$. We verify that $(\widetilde{y}_\xi, \widetilde{\Phi}_\xi)$ obeys:

$$\widetilde{y}' + A\widetilde{y} + A_e\widetilde{y} + \lambda_0\widetilde{y} = Bu, \quad \widetilde{y}(0) = \xi, \tag{5.2}$$

$$u = -B^*\widetilde{\Phi},$$

$$-\widetilde{\Phi}' + A\widetilde{\Phi} + A_{e,T}\widetilde{\Phi} + \lambda_0\widetilde{\Phi} = (I + 2\lambda_0\Pi)\widetilde{y}, \quad \widetilde{\Phi}(\infty) = 0. \tag{5.3}$$

The following proof relies on the successive use of regularity results for systems (5.2) and (5.3). In particular, one observes that because ξ belongs to (5.1) and because we have $\widetilde{\Phi}(0) = \Phi(0) = \Pi\xi$, the following compatibility condition is satisfied by $\widetilde{y}$:

$$\widetilde{y}(0) - PDu(0) = \xi + PDB^*\widetilde{\Phi}(0) = \xi + PDB^*\Pi\xi \in V_0^1(\Omega),$$

and it allows to invoke regularity results for system (5.2) when the boundary value is smooth [23, Theorem. 4.1 (iii) and (iv)]. Hence, in a first step, with the same bootstrap argument used to prove Corollary 4.3 in [21], we can prove that

$$\|\widetilde{\Phi}\|_{V^{7/2-\varepsilon,7/4-\varepsilon/2}(Q)} \leq C_0\|\xi\|_{V_n^{1/2-\varepsilon}(\Omega)} \quad \forall\varepsilon \in \left]0, 1/4\right[, \tag{5.4}$$

and in a second step, we obtain the desired result from (5.4) with the following calculations:

$$\begin{aligned}\|\widetilde{\Phi}(0)\|_{V_0^3(\Omega)} &\leq C_1\|\widetilde{\Phi}\|_{W(V_0^4(\Omega),V_0^2(\Omega))} \leq C_2\|\widetilde{\Phi}\|_{V^{4,2}(Q)} \leq C_3\|\widetilde{y}\|_{V^{2,1}(Q)}\\ &\leq C_4(\|B^*\widetilde{\Phi}\|_{V^{3/2,3/4}(\Sigma)} + \|\xi\|_{V_n^1(\Omega)})\\ &\leq C_5(\|\widetilde{\Phi}\|_{V^{7/2-\varepsilon,7/4-\varepsilon/2}(Q)} + \|\xi\|_{V_n^1(\Omega)}).\end{aligned}$$

The above calculations rely on the successive use of the continuous embeddings $W(V_0^4(\Omega), V_0^2(\Omega)) \hookrightarrow C_b(V_0^3(\Omega))$ and $V^{4,2}(Q) \hookrightarrow W(V_0^4(\Omega), V_0^2(\Omega))$, of regularity results [21, Lem. 8.5] for (5.2) and [23, Theorem. 4.1 (iv)] for (5.3), and of the estimate $\|B^*\widetilde{\Phi}\|_{V^{3/2,3/4}(\Sigma)} \leq C\|\widetilde{\Phi}\|_{V^{7/2-\varepsilon,7/4-\varepsilon/2}(Q)}$. This last estimate is an easy consequence of $B^* \in \mathcal{L}(V_0^{7/2-\varepsilon}(\Omega), V^{3/2}(\Gamma)) \cap \mathcal{L}(V_0^{3/2+\varepsilon}(\Omega), V^0(\Gamma))$ and of the continuous embedding $V^{7/2-\varepsilon,7/4-\varepsilon/2}(Q) \hookrightarrow L^2(V_0^{7/2-\varepsilon}(\Omega)) \cap H^{3/4}(V_0^{3/2+\varepsilon}(\Omega))$.

Remark 11. *In the proof of Lemma 2, the assumptions Ω of class C^4 and $z_e \in V^3(\Omega)$ are needed to obtain the estimate $\|\widetilde{\Phi}\|_{V^{4,2}(Q)} \leq C\|\widetilde{y}\|_{V^{2,1}(Q)}$. Indeed, the proof of [21, Lem. 8.5] relies on the estimate $\|\Phi\|_{V^4(\Omega)} \leq C\|f\|_{V_n^2(\Omega)}$ for the solution to*

$$\lambda_0\Phi + A\Phi + A_{e,T}\Phi = f \in V_n^2(\Omega),$$

[21, Lem. 8.4] which requires the assumptions Ω of class C^4 and $z_e \in V^3(\Omega)$.

Lemma 3. *The mapping $I + PD(B^*\Pi)$ is an isomorphism from*

$$\left\{ \xi \in V_n^2(\Omega) \mid \xi + PD(B^*\Pi)\xi \in V_0^2(\Omega) \right\} \tag{5.5}$$

onto $V_0^2(\Omega)$.

Proof. Since $I + PD(B^*\Pi)$ obviously defines a continuous operator from (5.5) onto $V_0^2(\Omega)$, the main difficulty is to prove that $(I + PD(B^*\Pi))^{-1}$ is well-defined as a continuous operator from $V_0^2(\Omega)$ onto (5.5). Let us first show that $(I+PD(B^*\Pi))^{-1} \in \mathcal{L}(V_n^0(\Omega))$. From $\Pi \in \mathcal{L}(V_n^0(\Omega), V_0^2(\Omega))$, (2.6) with $s = 1/2$, (2.3) with $s = 1$ and (2.1) with $s = 1$ we deduce that

$$PD(B^*\Pi) \in \mathcal{L}(V_n^0(\Omega), V_n^1(\Omega)). \tag{5.6}$$

As a consequence, since $V_n^1(\Omega)$ is compactly embedded in $V_n^0(\Omega)$, the operator $I + PD(B^*\Pi)$ is a compact perturbation of the identity and $I + PD(B^*\Pi) : V_n^0(\Omega) \longrightarrow$

$V_n^0(\Omega)$ is an isomorphism, if and only if, it is injective. Hence, we suppose that $\xi \in V_n^0(\Omega)$ satisfies $\xi + PD(B^*\Pi)\xi = 0$, and it remains to show that we necessary have $\xi = 0$. By making the inner product of $\xi + PD(B^*\Pi)\xi = 0$ by $(\lambda_0 + A + A_{e,T})\Pi\xi$ we first obtain:

$$((A + A_{e,T})\Pi\xi|\xi)_{V_n^0(\Omega)} + \lambda_0(\Pi\xi|\xi)_{V_n^0(\Omega)} + \|(B^*\Pi)\xi\|^2_{V^0(\Gamma)} = 0. \tag{5.7}$$

Moreover, since from (3.4) with $\zeta = \xi$ we have

$$((A + A_{e,T})\Pi\xi|\xi) + \frac{1}{2}\|(B^*\Pi)\xi\|^2_{V^0(\Gamma)} = \frac{1}{2}\|\xi\|^2_{V_n^0(\Omega)},$$

we can rewrite (5.7) as follows:

$$\frac{1}{2}\|\xi\|^2_{V_n^0(\Omega)} + \lambda_0(\Pi\xi|\xi)_{V_n^0(\Omega)} + \frac{1}{2}\|(B^*\Pi)\xi\|^2_{V^0(\Gamma)} = 0.$$

Hence, by recalling that Π is nonnegative, it follows that $\xi = 0$, which proves that $I + PD(B^*\Pi)$ is an isomorphism from $V_n^0(\Omega)$ onto $V_n^0(\Omega)$. Now, it remains to prove that $(I + PD(B^*\Pi))^{-1}$ is also a continuous operator from $V_0^2(\Omega)$ onto (5.5). Assume that $\xi \in V_n^0(\Omega)$ and $f \in V_0^2(\Omega)$ satisfy:

$$\xi + PD(B^*\Pi)\xi = f \in V_0^2(\Omega). \tag{5.8}$$

Since $(I + PD(B^*\Pi))^{-1} \in \mathcal{L}(V_n^0(\Omega))$ we have:

$$\|\xi\|_{V_n^0(\Omega)} \leq C\|f\|_{V_n^0(\Omega)}. \tag{5.9}$$

Moreover, with (5.6), (5.8) and (5.9) we obtain that ξ belongs to (5.1) from the following calculations:

$$\begin{aligned}\|\xi\|_{V_n^1(\Omega)} &\leq \|f\|_{V_0^1(\Omega)} + \|PD(B^*\Pi)\xi\|_{V_n^1(\Omega)} \leq \|f\|_{V_0^1(\Omega)} + C_1\|\xi\|_{V_n^0(\Omega)} \\ &\leq C_2\|f\|_{V_0^1(\Omega)}.\end{aligned} \tag{5.10}$$

Next, invoking lemma 2, from (2.6) with $s = 3/2$, from (2.3) with $s = 2$ and from (2.1) with $s = 2$, we deduce that $PD(B^*\Pi)$ is continuous from (5.1) onto $V_n^2(\Omega)$. Thus, from (5.8) and (5.10) it follows that ξ belongs to (5.5) and

$$\|\xi\|_{V_n^2(\Omega)} \leq \|f\|_{V_0^2(\Omega)} + \|PD(B^*\Pi)\xi\|_{V_n^2(\Omega)} \leq \|f\|_{V_0^2(\Omega)} + C_3\|\xi\|_{V_n^1(\Omega)} \leq C_4\|f\|_{V_0^2(\Omega)}.$$

THEOREM 9. *The unbounded operator* $(\mathcal{D}(A_\Pi), A_\Pi)$ *obeys*

$$\mathcal{D}(A_\Pi) = \left\{ \xi \in V_n^2(\Omega) \mid \xi + PD(B^*\Pi)\xi \in V_0^2(\Omega) \right\}.$$

Proof. According to (3.8), we have $\xi \in \mathcal{D}(A_\Pi)$, if and only if,

$$\lambda_0\xi + A\xi + A_e\xi + B(B^*\Pi)\xi \in V_n^0(\Omega),$$

which, with $B = (\lambda_0 + A + A_e)PD$, is equivalent to

$$(\lambda_0 + A + A_e)(\xi + PD(B^*\Pi)\xi) \in V_n^0(\Omega). \tag{5.11}$$

Thus, with $\mathcal{D}(\lambda_0 + A + A_e) = V_0^2(\Omega)$ and by Lemma 3, (5.11) means that ξ belongs to (5.5).

REMARK 12. *(i) Theorem 9 relies on the regularity property of* Π *stated in Lemma 2, which itself relies on the assumption* Ω *of class* C^4*, see Remark 11. Without any geometrical smoothness assumption on* Ω *the equality* $\mathcal{D}(A + A_e) = V_0^2(\Omega)$*, as well as the ismorphism property of* $I + PD(B^*\Pi)$ *stated in Lemma 3, may be lost, and the only characterization that can be deduced from (5.11) is:*

$$\mathcal{D}(A_\Pi) = \left\{ \xi \in V_n^0(\Omega) \mid \xi + PD(B^*\Pi)\xi \in \mathcal{D}(A + A_e) \right\}.$$

For instance, in the case where Ω *is a convex polyhedral domain Lemma 3 cannot be obtained. However, we still have* $\mathcal{D}(A + A_e) = V_0^2(\Omega)$ *and the boundedness of* Π *from* $V_n^0(\Omega)$ *into* $V_0^2(\Omega)$ *remains valid. Hence, starting from (5.8), with a bootstrap argument one can prove the following inclusion for all* $\varepsilon > 0$*:*

$$\mathcal{D}(A_\Pi) \subset \left\{ \xi \in V_n^{1-\varepsilon}(\Omega) \mid \xi + PD(B^*\Pi)\xi \in V_0^2(\Omega) \right\}.$$

One have $\varepsilon > 0$ *in the above inclusion because (2.1) is now only valid for* $s \in [0, 1[$*. It is due to the nonhomogeneous Neumann condition on a polyhedral boundary, which shall be considered for the Neumann problem related to P [14, Chap.III, Lem 1.2]. Notice that in the case where* Ω *is of class* C^2*, one can choose* $\varepsilon = 0$*.*
(ii) A characterization of the domain of a closed-loop operator associated with a Riccati-based feedback law is also given in [4, Prop. 6.1], in the case of a tangential control. However the geometrical smoothness assumption which is made there is not clear. In particular, it seems reasonable that the inclusion $\mathcal{D}(A^{5/4-\varepsilon_0/2}) \subset \mathbf{H}^{5/2-\varepsilon_0}(\Omega)$ *which is claimed in [4, Prop. 6.1, (6.7)] requires* Ω *of class* C^3*, and that the regularity theory invoked in [4, Prop. 6.1, step 5 and step 6] requires* Ω *of class* C^4*.*

COROLLARY 6. *The unbounded operator* $(\mathcal{D}(A_\Pi), A_\Pi)$ *obeys*

$$\mathcal{D}({A_\Pi}^{s/2}) = \left\{ \xi \in V_n^s(\Omega) \mid \xi + PD(B^*\Pi)\xi \in V_0^s(\Omega) \right\} \quad \forall s \in [0, 2]. \tag{5.12}$$

Proof. According to the first statement of (4.3), it suffices to prove that for $s \in]0, 2[$ the interpolation space $[\mathcal{D}(A_\Pi), V_n^0(\Omega)]_{1-s/2}$ is equal to the space defined at the right of equality (5.12). Since Lemma 3 and its proof state that $I + PD(B^*\Pi)$ is an isomorphism from $\mathcal{D}(A_\Pi)$ onto $V_0^2(\Omega)$, as well as from $V_n^0(\Omega)$ onto itsef, from the interpolation theorem [8, Chap.1, Thm. 4.1] we deduce that if $\xi \in [\mathcal{D}(A_\Pi), V_n^0(\Omega)]_{1-s/2}$ then $\xi + PD(B^*\Pi)\xi \in [V_0^2(\Omega), V_n^0(\Omega)]_{1-s/2}$ and conversely. As a consequence, since $[V_0^2(\Omega), V_n^0(\Omega)]_{1-s/2} = V_0^s(\Omega)$ we obtain that the interpolation space $[\mathcal{D}(A_\Pi), V_n^0(\Omega)]_{1-s/2}$ is exactly the space of functions $\xi \in V_n^0(\Omega)$ satisfying $\xi + PD(B^*\Pi)\xi \in V_0^s(\Omega)$. Finally, we conclude by observing that $\xi + PD(B^*\Pi)\xi \in V_0^s(\Omega)$ implies $\xi \in V_n^s(\Omega)$ because $(I + PDB^*\Pi)^{-1} \in \mathcal{L}(V_0^s(\Omega), V_n^s(\Omega))$. Indeed, it follows from $(I + PD(B^*\Pi))^{-1} \in \mathcal{L}(V_0^2(\Omega), V_n^2(\Omega)) \cap \mathcal{L}(V_n^0(\Omega))$ with and interpolation argument.

REMARK 13. *When* $s \in [0, 1/2[$ *we have* $V_0^s(\Omega) = V_n^s(\Omega)$ *and the condition* $\xi + PD(B^*\Pi)\xi \in V_0^s(\Omega)$ *in (5.12) is always satisfied. As a consequence, when* $s \in [0, 1/2[$ *we have* $\mathcal{D}(A_\Pi^{s/2}) = V_n^s(\Omega)$*.*

6. Stabilization of the nonlinear equation. Let us recall that $d=2$ or $d=3$ is the dimension of the geometrical domain Ω and let us fix $s\in[\frac{d-2}{2},1]$. The goal of the present section is to prove a local stabilization result for the following nonlinear system:

$$y'+A_\Pi y+N_\Pi(y)=0,\quad y(0)=y_0\in\mathcal{D}(A_\Pi^{s/2}). \tag{6.1}$$

We recall that, in the above setting, the nonlinear mapping $N_\Pi(\cdot)$ is defined by:

$$N_\Pi:\mathcal{D}(A_\Pi^{1/2})\longrightarrow V_0^{-1}(\Omega),\quad N_\Pi(\xi)=N\big(\xi-(I-P)D(B^*\Pi)\xi\big), \tag{6.2}$$

and that $N(\cdot)$ is given by (2.11). First, we give a boundedness property for the linear operator $I-(I-P)D(B^*\Pi)$ appearing in (6.2).

Lemma 4. *The linear operator $I-(I-P)D(B^*\Pi)$ obeys:*

$$I-(I-P)D(B^*\Pi)\in\mathcal{L}(\mathcal{D}(A_\Pi^{\theta/2}),V^\theta(\Omega)),\quad\forall\theta\in[0,2]. \tag{6.3}$$

Proof. According to Theorem 6 and Lemma 2, the operator Π is bounded from $V_n^0(\Omega)$ into $V_0^2(\Omega)$ as well as from $\mathcal{D}(A_\Pi^{1/2})$ into $V_0^3(\Omega)$. Hence, with an interpolation argument, we obtain that Π also belongs to $\mathcal{L}(\mathcal{D}(A_\Pi^{s/2}),V_0^{2+s}(\Omega))$ for all $s\in[0,1]$, and from (2.1), (2.3) and (2.6) we deduce that:

$$(I-P)D(B^*\Pi)\in\mathcal{L}(\mathcal{D}(A_\Pi^{s/2}),V^{1+s}(\Omega))\quad\forall s\in[0,1]. \tag{6.4}$$

Finally, because for all $\theta\in[0,2]$ we have the continuous embedding $\mathcal{D}(A_\Pi^{\theta/2})\hookrightarrow V^\theta(\Omega)$, which exactly means that the identity operator I is bounded from $\mathcal{D}(A_\Pi^{\theta/2})$ into $V^\theta(\Omega)$, we obtain (6.3) from (6.4).

The following lemmas deal with boundedness and Lipschitz properties of $N_\Pi(\cdot)$.

Lemma 5. *Let $N_\Pi(\cdot)$ be the nonlinear mapping defined by (6.2).*
If $s\in[\frac{d-2}{2},1]$, then for all $\xi\in\mathcal{D}(A_\Pi^{1/2+s/2})$ the following estimate holds:

$$\|N_\Pi(\xi)\|_{V_0^{s-1}(\Omega)}\le C\|\xi\|_{\mathcal{D}(A_\Pi^{s/2})}\|\xi\|_{\mathcal{D}(A_\Pi^{1/2+s/2})}. \tag{6.5}$$

If $s\in[\frac{d-2}{2},1]$ and $s\neq0$, then for all $(\xi,\zeta)\in\mathcal{D}(A_\Pi^{1/2+s/2})\times\mathcal{D}(A_\Pi^{1/2+s/2})$ the following estimate holds:

$$\|N_\Pi(\xi)-N_\Pi(\zeta)\|_{V_0^{s-1}(\Omega)}\le C(\|\xi-\zeta\|_{\mathcal{D}(A_\Pi^{s/2})}\|\xi\|_{\mathcal{D}(A_\Pi^{1/2+s/2})}+\|\zeta\|_{\mathcal{D}(A_\Pi^{s/2})}\|\xi-\zeta\|_{\mathcal{D}(A_\Pi^{1/2+s/2})}). \tag{6.6}$$

If $d=2$ and $s=0$, then for all $(\xi,\zeta)\in\mathcal{D}(A_\Pi^{1/4})\times\mathcal{D}(A_\Pi^{1/4})$ the following estimate holds:

$$\|N_\Pi(\xi)-N_\Pi(\zeta)\|_{V_0^{-1}(\Omega)}\le C\|\xi-\zeta\|_{\mathcal{D}(A_\Pi^{1/4})}(\|\xi\|_{\mathcal{D}(A_\Pi^{1/4})}+\|\zeta\|_{\mathcal{D}(A_\Pi^{1/4})}). \tag{6.7}$$

Proof. If $s>0$, estimate (6.5) is an easy consequence of (2.9) and (6.3). Indeed, (2.9) with $(s_1,s_2,s_3)=(s,s,1-s)$ yields:

$$\|N_\Pi(\xi)\|_{V_0^{s-1}(\Omega)}\le C\|\xi-(I-P)D(B^*\Pi)\xi\|_{V^s(\Omega))}\|\xi-(I-P)D(B^*\Pi)\xi\|_{V^{s+1}(\Omega)}, \tag{6.8}$$

and we conclude by using (6.3) with $\theta = s$ and $\theta = s+1$. Estimate (6.6) follows in a similar way. Next, if $s = 0$, from (2.10) for $(s_1, s_2, s_3) = (1/2, 1/2, 0)$ and (6.3) we obtain:

$$\|N_\Pi(\xi)\|_{V_0^{-1}(\Omega)} \le C\|\xi\|^2_{\mathcal{D}(A_\Pi^{1/4})}, \tag{6.9}$$

and the interpolation inequality $\|\cdot\|^2_{\mathcal{D}(A_\Pi^{1/4})} \le C\|\cdot\|_{V_n^0(\Omega)}\|\cdot\|_{\mathcal{D}(A_\Pi^{1/2})}$ gives (6.5) for $s = 0$. Finally, estimate (6.7) follows from (2.10) for $(s_1, s_2, s_3) = (1/4, 1/4, 0)$ and (6.3).

LEMMA 6. *Let $N_\Pi(\cdot)$ be the nonlinear mapping defined by (6.2) and let $s \in [\frac{d-2}{2}, 1]$. Then for all z, z_1 and z_2 in $W(\mathcal{D}(A_\Pi^{1/2+s/2}), V_0^{s-1}(\Omega))$ the following estimates hold:*

$$\|N_\Pi(z)\|_{L^2(V_0^{s-1}(\Omega))} \le C\|z\|^2_{W(\mathcal{D}(A_\Pi^{1/2+s/2}), V_0^{s-1}(\Omega))}, \tag{6.10}$$

and

$$\begin{aligned}\|N_\Pi(z_1) - N_\Pi(z_2)\|_{L^2(V_0^{s-1}(\Omega))} &\le C\|z_1 - z_2\|_{W(\mathcal{D}(A_\Pi^{1/2+s/2}), V_0^{s-1}(\Omega))} \\ &\times(\|z_1\|_{W(\mathcal{D}(A_\Pi^{1/2+s/2}), V_0^{s-1}(\Omega))} + \|z_2\|_{W(\mathcal{D}(A_\Pi^{1/2+s/2}), V_0^{s-1}(\Omega))})\end{aligned} \tag{6.11}$$

Proof. From (6.5) we deduce that:

$$\|N_\Pi(z)\|_{L^2(V_0^{s-1}(\Omega))} \le C\|z\|_{C_b(\mathcal{D}(A_\Pi^{s/2}))}\|z\|_{L^2(\mathcal{D}(A_\Pi^{1/2+s/2})}, \tag{6.12}$$

and (6.10) follows from $W(\mathcal{D}(A_\Pi^{1/2+s/2}), V_0^{s-1}(\Omega)) \hookrightarrow C_b(\mathcal{D}(A_\Pi^{s/2}))$. Next, if $s > 0$, from (6.6) we have:

$$\begin{aligned}\|N_\Pi(z_1) - N_\Pi(z_2)\|_{L^2(V_0^{s-1}(\Omega))} \le C(&\|z_1 - z_2\|_{C_b(\mathcal{D}(A_\Pi^{s/2})}\|z_2\|_{L^2(\mathcal{D}(A_\Pi^{1/2+s/2})} \\ &+ \|z_1\|_{C_b(\mathcal{D}(A_\Pi^{s/2}))}\|z_1 - z_2\|_{L^2(\mathcal{D}(A_\Pi^{1/2+s/2})}),\end{aligned}$$

and (6.11) follows from $W(\mathcal{D}(A_\Pi^{1/2+s/2}), V_0^{s-1}(\Omega)) \hookrightarrow C_b(\mathcal{D}(A_\Pi^{s/2}))$. Finally, if $d = 2$ and $s = 0$, from (6.7) we deduce that:

$$\|N_\Pi(z_1) - N_\Pi(z_2)\|_{L^2(V_0^{-1}(\Omega))} \le C(\|z_1 - z_2\|_{L^4(\mathcal{D}(A_\Pi^{1/4}))}(\|z_1\|_{L^4(\mathcal{D}(A_\Pi^{1/4}))} + \|z_2\|_{L^4(\mathcal{D}(A_\Pi^{1/4}))}),$$

and (6.11) follows from the continuous embeddings $W(\mathcal{D}(A_\Pi^{1/2}), V_0^{-1}(\Omega)) \hookrightarrow H^{1/4}(\mathcal{D}(A_\Pi^{1/4})) \hookrightarrow L^4(\mathcal{D}(A_\Pi^{1/4}))$.

We are now in position to give a first existence result for system (6.1).

THEOREM 10. *Let $s \in [\frac{d-2}{2}, 1]$ and $y_0 \in \mathcal{D}(A_\Pi^{s/2})$. There exist $\rho_0 > 0$ and $\mu_0 > 0$ such that, if $\delta \in (0, \mu_0)$ and*

$$\|y_0\|_{\Pi,s} < \rho_0\delta, \tag{6.13}$$

system (6.1) admits a unique solution in the set

$$\mathcal{S}^s_\delta = \left\{ y \in W(\mathcal{D}(A_\Pi^{1/2+s/2}), V_0^{s-1}(\Omega)) \mid \|y\|_{W(\mathcal{D}(A_\Pi^{1/2+s/2}), V_0^{s-1}(\Omega))} \le \delta \right\}. \tag{6.14}$$

Proof. The proof is based on a fixed point argument. Let us consider the mapping Ψ defined on $W(\mathcal{D}(A_\Pi^{1/2+s/2}), V_0^{s-1}(\Omega))$ by $\Psi(z) = y_z$ where y_z is the unique solution to the following linear system:

$$y' + A_\Pi y + N_\Pi(z) = 0, \quad y(0) = y_0. \tag{6.15}$$

Let us seek $\rho_0 > 0$ and $\mu_0 > 0$ such that, for every y_0 obeying (6.13) with $\delta \in (0, \mu_0)$, Ψ is a contraction in $\mathcal{S}_\delta^s$. First, since $y_z = \Psi(z)$ is solution to (6.15), the successive use of (4.6), (4.10) and (6.10) provides two constants $C_0 > 0$ and $C_1 > 0$ such that:

$$\|\Psi(z)\|_{W(\mathcal{D}(A_\Pi^{1/2+s/2}), V_0^{s-1}(\Omega))} \le C_0(C_1 \|z\|^2_{W(\mathcal{D}(A_\Pi^{1/2+s/2}), V_0^{s-1}(\Omega))} + \|y_0\|_{\Pi,s}).$$

As a consequence, because z belongs to $\mathcal{S}_\delta^s$ and y_0 obeys (6.13), we have:

$$\|\Psi(z)\|_{W(\mathcal{D}(A_\Pi^{1/2+s/2}), V_0^{s-1}(\Omega))} \le C_0(C_1\mu_0 + \rho_0)\delta. \tag{6.16}$$

Next, we verify that $y = \Psi(z_1) - \Psi(z_2)$ is solution to

$$y' + A_\Pi y + N_\Pi(z_1) - N_\Pi(z_2) = 0, \quad y(0) = 0,$$

and for z_1 and z_2 in $\mathcal{S}_\delta^s$, the successive use of (4.6) and (6.11) provides a constant $C_2 > 0$ such that:

$$\|\Psi(z_1) - \Psi(z_2)\|_{W(\mathcal{D}(A_\Pi^{1/2+s/2}), V_0^{s-1}(\Omega))} \le 2\mu_0 C_2 \|z_1 - z_2\|_{W(\mathcal{D}(A_\Pi^{1/2+s/2}), V_0^{s-1}(\Omega))}.$$

Finally, by choosing $\mu_0 = \min(\frac{1}{2C_0C_1}, \frac{1}{4C_2})$ and $\rho_0 < \frac{1}{2C_0}$, the above estimate and (6.16) mean that Ψ is a contraction in $\mathcal{S}_\delta^s$ and that (6.1) admits a unique solution in $\mathcal{S}_\delta^s$.

REMARK 14. *We shall underline that Theorem 10 critically relies on the assumption Ω of class C^4, because such a geometrical assumption is needed to prove the continuous embedding $\mathcal{D}(A_\Pi^\theta) \hookrightarrow V_n^\theta(\Omega)$, which is required to prove (6.3). Indeed, (6.3) and (6.8) give (6.10), which is the key tool in the proof of Thoerem 10 to prove that Ψ maps $\mathcal{S}_\delta^s$ into itself.*

Theorem 10 is not really satisfactory because it only guarantees the uniqueness of the solution in the neighborhood of zero $\mathcal{S}_\delta^s$. In order to obtain a larger uniqueness result, as well as the exponential decrease of the solution to (6.1), we need some a priori estimates which can be deduced from an adequate use of the inner product $(\cdot|\cdot)_{\Pi,s}$ given by (4.9). In the following lemma, we first give a useful estimate of $(N_\Pi(\cdot)|\cdot)_{\Pi,s}$.

LEMMA 7. *For all $s \in [\frac{d-2}{2}, 1]$, there exists a constant $C_s > 0$ such that the following estimate holds:*

$$(N_\Pi(\xi)|\xi)_{\Pi,s} \le C_s \|\xi\|_{\Pi,s} (A_\Pi \xi|\xi)_{\Pi,s} \quad \forall \xi \in \mathcal{D}(A_\Pi^{1/2+s/2}). \tag{6.17}$$

Proof. First, if $s > 0$, we recall that (4.8) for $\theta = 1$ gives:

$$\Pi^{(s)} = A_\Pi^{*s/2+1/2}\, \Pi\, A_\Pi^{s/2+1/2} \in \mathcal{L}(\mathcal{D}(A_\Pi^{1/2+s/2}), V_0^{1-s}(\Omega)), \tag{6.18}$$

and by recalling (6.2), from (2.9) for $(s_1, s_2, s_3) = (s, s, 1-s)$, (6.3) and (6.18) we obtain:

$$(N_\Pi(\xi)|\xi)_{\Pi,s} = |\langle N_\Pi(\xi)|\Pi^{(s)}\xi\rangle_{V_0^{s-1}(\Omega), V_0^{1-s}(\Omega)}| \le C\|\xi\|_{\mathcal{D}(A_\Pi^{s/2})} \|\xi\|^2_{\mathcal{D}(A_\Pi^{1/2+s/2})}, \tag{6.19}$$

for all $\xi \in \mathcal{D}(A_\Pi^{1/2+s/2})$. Hence, (6.17) follows from (4.10) and (4.11). Next, if $s=0$, we recall that (4.8) for $\theta = 1/2$ gives:

$$\Pi^{(0)} = A_\Pi^{*1/2}\, \Pi\, A_\Pi^{1/2} \in \mathcal{L}(\mathcal{D}(A_\Pi^{1/4}), V_0^{1/2}(\Omega)), \tag{6.20}$$

and by recalling (6.2), from (2.9) for $(s_1, s_2, s_3) = (1/2, 0, 1/2)$, (6.3) and (6.20) we obtain:

$$(N_\Pi(\xi)|\xi)_{\Pi,0} = |(N_\Pi(\xi)|\Pi^{(0)}\xi)| \le C\|\xi\|^2_{\mathcal{D}(A_\Pi^{1/4})}\|\xi\|_{\mathcal{D}(A_\Pi^{1/2})} \quad \forall \xi \in \mathcal{D}(A_\Pi^{1/2}).$$

Hence, the interpolation inequality $\|\cdot\|_{\mathcal{D}(A_\Pi^{1/4})} \le C\|\cdot\|^{1/2}_{V_n^0(\Omega)}\|\cdot\|^{1/2}_{\mathcal{D}(A_\Pi^{1/2})}$ yields (6.19) with $s=0$, and (6.17) for $s=0$ follows from (4.10) and (4.11).
We are now in position to prove that for y_0 small enough in $\mathcal{D}(A_\Pi^{s/2})$, system (6.1) admits a solution which is unique within the class of function belonging to $L^\infty_{loc}(\mathcal{D}(A_\Pi^{s/2})) \cap L^2_{loc}(\mathcal{D}(A_\Pi^{1/2+s/2}))$. The following theorem is based on a priori estimates which are obtained from the $(\cdot|\cdot)_{\Pi,s}$-product of the first equation in (6.1) by y, see (6.24) and (6.25) below.

THEOREM 11. *Let $s \in [\frac{d-2}{2}, 1]$. There exist $\rho_1 > 0$ and $\mu_1 > 0$ such that, if $\delta \in (0, \mu_1)$ and*

$$y_0 \in \mathcal{I}_\delta^s = \left\{ y \in \mathcal{D}(A_\Pi^{s/2}) \mid \|y\|_{\Pi,s} < \rho_1 \delta \right\}, \tag{6.21}$$

system (6.1) admits a solution y_{y_0} in the set $\mathcal{S}_\delta^s$ given by (6.14). Moreover, the solution y_{y_0} is unique within the class of functions belonging to $L^\infty_{loc}(\mathcal{D}(A_\Pi^{s/2})) \cap L^2_{loc}(\mathcal{D}(A_\Pi^{1/2+s/2}))$, the mapping $t \longmapsto \|y_{y_0}(t)\|^2_{\Pi,s}$ decreases to 0, and there exists $\sigma > 0$ such that:

$$\|y_{y_0}(t)\|_{\Pi,s} \le \|y_0\|_{\Pi,s} e^{-\sigma t} \quad \forall t \ge 0. \tag{6.22}$$

Proof. In a first step, let us prove that if $\|y_0\|_{\Pi,s} < \frac{1}{2C_s}$ for $C_s > 0$ given in (6.17), then there is $C_0 > 0$ such that every y solution to (6.1) in $L^\infty_{loc}(\mathcal{D}(A_\Pi^{s/2})) \cap L^2_{loc}(\mathcal{D}(A_\Pi^{1/2+s/2}))$ obeys (6.22) and

$$\|y\|_{W(\mathcal{D}(A_\Pi^{1/2+s/2}), V_0^{s-1}(\Omega))} \le C_0\|y_0\|_{\Pi,s}. \tag{6.23}$$

First, assume that $\|y_0\|_{\Pi,s} < \frac{1}{2C_s}$ and that y belongs to $L^\infty_{loc}(\mathcal{D}(A_\Pi^{s/2})) \cap L^2_{loc}(\mathcal{D}(A_\Pi^{1/2+s/2}))$ and obeys (6.1). From (6.5) we deduce that $N_\Pi(y)$ belongs to $L^2_{loc}(V_0^{s-1}(\Omega))$, and from (6.1) we obtain $y \in W_{loc}(\mathcal{D}(A_\Pi^{1/2+s/2}), V_0^{s-1}(\Omega))$. Thus, by multiplying the first equality in (6.1) by $\Pi^{(s)}y(t)$ we obtain:

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|^2_{\Pi,s} + (A_\Pi y(t)|y(t))_{\Pi,s} = (N_\Pi(y(t))|y(t))_{\Pi,s},$$

and from (6.17) we have:

$$\frac{d}{dt}\|y(t)\|^2_{\Pi,s} + 2(1 - C_s\|y(t)\|_{\Pi,s})(A_\Pi y(t)|y(t))_{\Pi,s} \le 0.$$

Hence, because $\|y_0\|_{\Pi,s} < \frac{1}{2C_s}$, the mapping $t \longmapsto \|y(t)\|_{\Pi,s}$ is a nonincreasing function with values less than $\frac{1}{2C_s}$ and we have:

$$\frac{d}{dt}\|y(t)\|^2_{\Pi,s} + (A_\Pi y(t)|y(t))_{\Pi,s} \leq 0. \tag{6.24}$$

As a consequence, (6.22) holds for a rate $\sigma > 0$ obeying $2\sigma\|\cdot\|^2_{\Pi,s} \leq (A_\Pi \cdot|\cdot)_{\Pi,s}$, and by integrating (6.24) over $(0,\infty)$ we also obtain the following estimate:

$$\int_0^\infty (A_\Pi y(t)|y(t))_{\Pi,s} dt \leq \|y_0\|^2_{\Pi,s}. \tag{6.25}$$

Finally, starting from the expression of y' given by (6.1), the use of (6.12), of (6.22) with (4.10) and of (6.25) with (4.11), yields $\|y'\|_{L^2(V_0^{s-1}(\Omega))} \leq C\|y_0\|^2_{\Pi,s}$, which provides a constant $C_0 > 0$ for which (6.23) is true.

In a second step, let us exhibit $\rho_1 > 0$ and $\mu_1 > 0$ for which the theorem is true. For $\rho_0 > 0$ and $\mu_0 > 0$ given in Theorem 10 we set $\rho_1 = \min(\rho_0, \frac{1}{C_0})$ and $\mu_1 = \min(\mu_0, \frac{1}{2\rho_1 C_s})$, and we assume that for $\delta \in (0,\mu_1)$ the initial datum y_0 belongs to $\mathcal{I}^s_\delta$ defined by (6.21). One a first hand, since $0 < \rho_1 \leq \rho_0$ and $0 < \mu_1 \leq \mu_0$ we have $\|y_0\|_{\Pi,s} \leq \rho_0\delta$ with $\delta \in (0,\mu_0)$, and by Theorem 10 system (2.22) admits a unique solution y_{y_0} in $\mathcal{S}^s_\delta$. One another hand, because $y_0 \in \mathcal{I}^s_\delta$ and $\mu_1 \leq \frac{1}{2\rho_1 C_s}$ imply $\|y_0\|_{\Pi,s} \leq \frac{1}{2C_s}$, then as it has been seen, if $y \in L^\infty_{loc}(\mathcal{D}(A_\Pi^{s/2})) \cap L^2_{loc}(\mathcal{D}(A_\Pi^{1/2+s/2}))$ is solution to (6.1) then it obeys (6.23). Hence, because $\rho_1 \leq \frac{1}{C_0}$ we have $C_0\|y_0\|_{\Pi,s} \leq \delta$ and (6.23) means that y belongs to $\mathcal{S}^s_\delta$. Finally, because the solution to (6.1) is unique in $\mathcal{S}^s_\delta$, y and y_{y_0} must coincide.

Finally, let us prove our main theorem.

Proof of Theorem 5

Let $s \in [\frac{d-2}{2}, 1]$, and let $\rho_1 > 0$ and $\mu_1 > 0$ be the ones given in Theorem 11. For $\delta \in (0,\mu_1)$ the condition $P(z_0 - z_e) \in \mathcal{I}^s_\delta$ ensures that system (6.1) where $y_0 = P(z_0 - z_e)$ admits a solution y in $W(\mathcal{D}(A_\Pi^{1/2+s/2}), V_0^{s-1}(\Omega))$, which is unique within the class of functions belonging to $L^\infty_{loc}(\mathcal{D}(A_\Pi^{s/2})) \cap L^2_{loc}(\mathcal{D}(A_\Pi^{1/2+s/2}))$. Thus, we verify that $\widetilde{y} = y - (I-P)D(B^*\Pi)y$ belongs to $V^{s+1,s/2+1/2}_{loc}(Q)$ and satisfies:

$$P\widetilde{y}' + AP\widetilde{y} + A_e P\widetilde{y} + N(\widetilde{y}) = -B(B^*\Pi)P\widetilde{y}, \qquad P\widetilde{y}(0) = P(z_0 - z_e), \tag{6.26}$$

$$(I-P)\widetilde{y} = -(I-P)D(B^*\Pi)P\widetilde{y}. \tag{6.27}$$

Conversely, if a function $\widetilde{y} \in V^{s+1,s/2+1/2}_{loc}(Q)$ obeys the above equations, then $P\widetilde{y}$ obeys (6.1) and belongs to $L^\infty_{loc}(\mathcal{D}(A_\Pi^{s/2})) \cap L^2_{loc}(\mathcal{D}(A_\Pi^{1/2+s/2}))$. Indeed, $\widetilde{y} \in V^{s+1,s/2+1/2}_{loc}(Q)$ implies $P\widetilde{y} \in H^{s/2+1/2}_{loc}(V^0_n(\Omega)) \cap L^2_{loc}(\mathcal{D}(A_\Pi^{1/2+s/2}))$. If $s > 0$, it guarantees that $P\widetilde{y} \in L^\infty_{loc}([\mathcal{D}(A_\Pi^{s/2+1/2}), V^0_n(\Omega)]_{1/(1+s)}) = L^\infty_{loc}(\mathcal{D}(A_\Pi^{s/2}))$, see [16, Def. 2.2], and if $s = 0$, (6.9) with (6.1) successively gives $N_\Pi(P\widetilde{y}) \in L^2_{loc}(V_0^{-1}(\Omega))$, $P\widetilde{y} \in W_{loc}(\mathcal{D}(A_\Pi^{1/2}), V_0^{-1}(\Omega))$ and $P\widetilde{y} \in L^\infty_{loc}(V^0_n(\Omega))$. As a consequence, since from (6.27) we have that $\widetilde{y}$ is entirely determined by its projected part $P\widetilde{y}$, the uniqueness of the solution to (6.1) implies the uniqueness of the solution to (6.26)-(6.27) in $V^{s+1,s/2+1/2}_{loc}(Q)$. Hence, according to Proposition 3 and Proposition 4, there exists $p \in H^{-1/2+s/2}_{loc}(\mathcal{H}^s(\Omega))$ such that $(z,r) = (z_e + \widetilde{y}, r_e + p)$ is the unique solution to (2.25)-(2.26)-(2.27)-(2.28) in

$$\big\{(z_e, r_e)\big\} + V^{s+1,s/2+1/2}_{loc}(Q) \times H^{-1/2+s/2}_{loc}(\mathcal{H}^s(\Omega)).$$

Moreover, since $P\widetilde{y} = P(z - z_e)$ is the solution to (6.1) given by Theorem 11, then $t \longmapsto \|P(z(t) - z_e)\|^2_{\Pi,s}$ decreases to 0 and (2.31) is true.

Next, it remains to prove that $(\widetilde{y}, p) = (z - z, r - r_e)$ belongs to the set $\mathcal{D}^s_\delta$ given by (2.30). Let us give an estimate of $(\widetilde{y}, p) = (z - z, r - r_e)$ in $V^{1+s,1/2+s/2}(Q) \times H^{-1/2+s/2}(\mathcal{H}^s(\Omega))$ in function of δ. First, because $y = P\widetilde{y}$ belongs to the set $\mathcal{S}^s_\delta$ given by (6.14) and because $W(\mathcal{D}(A_\Pi^{1/2+s/2}), V_0^{s-1}(\Omega)) \hookrightarrow V^{1+s,1/2+s/2}(Q)$, from (6.3) we deduce that $\widetilde{y} = y - (I - P)D(B^*\Pi)y$ belongs to $V^{1+s,1/2+s/2}(Q)$ and that there is $c_1 > 0$ such that it obeys:

$$\|\widetilde{y}\|_{V^{1+s,1/2+s/2}(Q)} \leq c_1\delta. \tag{6.28}$$

Hence, it remains to estimate $p = r - r_e$ in $H^{-1/2+s/2}(\mathcal{H}^s(\Omega))$. First, subtracting (1.1) from (2.25) yields:

$$\nabla(r - r_e) = -\partial_t\widetilde{y} + \nu\Delta\widetilde{y} - (\widetilde{y}\cdot\nabla)z_e - (z_e\cdot\nabla)\widetilde{y} - (\widetilde{y}\cdot\nabla)\widetilde{y}, \tag{6.29}$$

and since there is $c_2 > 0$ such that:

$$\|r - r_e\|_{H^{-1/2+s/2}(\mathcal{H}^s(\Omega))} \leq c_2\|\nabla(r - r_e)\|_{H^{-1/2+s/2}(\mathbf{H}^{s-1}(\Omega))}, \tag{6.30}$$

it suffices to estimate each term at the right of equality (6.29) to obtain an estimate of $r - r_e$ in $H^{-1/2+s/2}(\mathcal{H}^s(\Omega))$. In a first step, we invoke the continuous embedding of $H^{-1/2+s/2}(V^0(\Omega))$ into $H^{-1/2+s/2}(\mathbf{H}^{s-1}(\Omega))$, as well as the boundedness of ∂_t from $H^{1/2+s/2}(V^0(\Omega))$ into $H^{-1/2+s/2}(V^0(\Omega))$, to obtain $c_3 > 0$ such that:

$$\|\partial_t\widetilde{y}\|_{H^{-1/2+s/2}(\mathbf{H}^{s-1}(\Omega))} \leq c_3\|\widetilde{y}\|_{H^{1/2+s/2}(V^0(\Omega))}. \tag{6.31}$$

In a second step, we invoke the continuous embedding of $L^2(\mathbf{H}^{s-1}(\Omega))$ into $H^{-1/2+s/2}(\mathbf{H}^{s-1}(\Omega))$ and the boundedness of $\nu\Delta - (\nabla z_e) - (z_e\cdot\nabla)$ from $L^2(V^{s+1}(\Omega)))$ into $L^2(\mathbf{H}^{s-1}(\Omega))$ to obtain $c_4 > 0$ such that:

$$\|\nu\Delta\widetilde{y} - (\widetilde{y}\cdot\nabla)z_e - (z_e\cdot\nabla)\widetilde{y}\|_{H^{-1/2+s/2}(\mathbf{H}^{s-1}(\Omega)))} \leq c_4\|\widetilde{y}\|_{L^2(V^{s+1}(\Omega)))}. \tag{6.32}$$

Hence, if in a third step we prove the existence of $c_5 > 0$ such that:

$$\|(\widetilde{y}\cdot\nabla)\widetilde{y}\|_{H^{-1/2+s/2}(\mathbf{H}^{s-1}(\Omega)))} \leq c_5\|\widetilde{y}\|^2_{V^{1+s,1/2+s/2}(Q)}, \tag{6.33}$$

then (6.29) with (6.30), (6.31), (6.32) and (6.33) will yield:

$$\|r - r_e\|_{H^{-1/2+s/2}(\mathcal{H}^s(\Omega))} \leq c_2(c_3 + c_4)\|\widetilde{y}\|_{V^{1+s,1/2+s/2}(Q)} + c_2c_5\|\widetilde{y}\|^2_{V^{1+s,1/2+s/2}(Q)}. \tag{6.34}$$

To obtain (6.33), we observe that for every element v in $H^{1/2-s/2}(\mathbf{H}_0^{1-s}(\Omega))$, the use of (2.9) with $(s_1, s_2, s_3) = (s, s, 1 - s)$ if $s > 0$, or (2.10) with $(s_1, s_2, s_3) = (1/2, 1/2, 0)$ if $s = 0$, provides $c'_1 > 0$ such that:

$$b(\widetilde{y}(t), \widetilde{y}(t), v(t)) \leq c'_1\|\widetilde{y}(t)\|_{V^s(\Omega)}\|\widetilde{y}(t)\|_{V^{1+s}(\Omega)}\|v(t)\|_{\mathbf{H}_0^{1-s}(\Omega)},$$

from which an estimation in time gives $c'_2 > 0$ such that:

$$\int_0^\infty b(\widetilde{y}(t), \widetilde{y}(t), v(t))dt \leq c'_2\|\widetilde{y}\|_{H^{1/2+s/2}(V^s(\Omega))}\|\widetilde{y}(t)\|_{L^2(V^{1+s}(\Omega))}\|v(t)\|_{H^{1/2-s/2}(\mathbf{H}_0^{1-s}(\Omega))}.$$

Hence, because $H^{-1/2+s/2}(\mathbf{H}^{s-1}(\Omega)) = [H^{1/2-s/2}(\mathbf{H}_0^{1-s}(\Omega))]'$, taking the sup over all v in the unit sphere of $H^{1/2-s/2}(\mathbf{H}_0^{1-s}(\Omega)))$ yields:

$$\|(\widetilde{y}\cdot\nabla)\widetilde{y}\|_{H^{-1/2+s/2}(\mathbf{H}^{s-1}(\Omega)))} \leq c_4'\|\widetilde{y}\|_{H^{1/2+s/2}(V^s(\Omega))}\|\widetilde{y}\|_{L^2(V^{1+s}(\Omega))},$$

from which (6.33) follows. Next, by taking into account that $\widetilde{y}$ obeys (6.28), from (6.34) we deduce that:

$$\|r-r_e\|_{H^{-1/2+s/2}(\mathcal{H}^s(\Omega))} \leq c_1c_2(c_3+c_4)\delta + c_1c_2c_5\delta^2,$$

and by setting $c_6 = \max(1, c_1, c_1c_2(c_3+c_4), c_1c_2c_5)$ we finally obtain:

$$\|z-z_e\|_{V^{1+s,1/2+s/2}(Q)} \leq c_6\delta \ \text{ and } \ \|r-r_e\|_{H^{-1/2+s/2}(\mathcal{H}^s(\Omega))} \leq c_6\delta(1+\delta). \tag{6.35}$$

To conclude, it suffices to choose $\mu = \mu_1$ and $\rho = \rho_1/c_6$ in the definition of $\mathcal{W}_\delta^s$ given by (2.29). Indeed, if $P(z_0-z_e)$ belongs to $\mathcal{W}_\delta^s$ for $\delta \in (0,\mu)$, then $P(z_0-z_e)$ belongs to $\mathcal{I}^s_{\delta/c_6}$. Hence, since $c_6 \geq 1$ we have $\delta/c_6 \in (0,\mu) = (0,\mu_1)$, and it yields (6.35) with δ/c_6 instead of δ. Finally, $\delta/c_6 \leq \delta$ ensures that $(z-z_e, r-r_e) \in \mathcal{D}_\delta^s$ given by (2.30). □

REMARK 15. *Since the solution $z - z_e = \widetilde{y}$ obeys (6.27), from (6.4) we deduce that:*

$$\|(I-P)(z(t)-z_e)\|_{V^{1+s}(\Omega)} \leq C\|P(z(t)-z_e)\|_{\Pi,s}, \quad t \geq 0.$$

As a consequence, (2.24) yields the following exponential decay:

$$\|(I-P)(z(t)-z_e)\|_{V^{1+s}(\Omega)} + \|P(z(t)-z_e)\|_{V_n^s(\Omega)} \leq C\|Py_0\|_{\Pi,s}e^{-\sigma t}, \quad t \geq 0.$$

7. Feedback control localized in a part of the boundary. In the previous sections, we have considered a boundary control acting on the whole boundary Γ. Nevertheless, it is possible to treat the case of a boundary control which is localized in an open subset of Γ. By following the idea of [21] we introduce a weight function $m \in C^2(\Gamma)$ with values in $[0,1]$, with support in $\Gamma_m \subset \Gamma$ and equal to 1 in Γ_1, where Γ_1 is an open subset of Γ_m. We define the operator $M \in \mathcal{L}(\mathbf{L}^2(\Gamma); V^0(\Gamma))$ as follows:

$$M : v \longmapsto m\bigg(v - \Big(\int_\Gamma m\Big)^{-1}\Big(\int_\Gamma mv\cdot n\Big)n\bigg) \quad \forall v \in \mathbf{L}^2(\Gamma),$$

and we now consider the following evolution system with a nonhomogeneous Dirichlet boundary condition localized on Γ_m:

$$y' + Ay + A_ey = BMu \ \in \mathcal{D}(A + A_{e,T})', \quad y(0) = y_0 \in V_n^0(\Omega). \tag{7.1}$$

From a null controllability result by means of a distributed control which is stated in [12], we can obtain a null controllability result for a control localized on Γ_1 by using an extension of the domain Ω. Then for $\mathcal{J}$ defined by (3.3), the minimization problem:

$$\inf\Big\{\ \mathcal{J}(y,u)\ \mid\ (y,u) \in W(V_n^0(\Omega), V_0^{-2}(\Omega)) \times L^2(V^0(\Gamma)) \ \text{ satisfies } \ (7.1)\Big\}$$

admits a unique solution (y_{y_0}, u_{y_0}), and y_{y_0} is the solution to the closed-loop system:

$$y' + Ay + A_e y + BM^2(B^*\Pi_M)y = 0, \quad y(0) = y_0 \in V_n^0(\Omega),$$

where Π_M is a unique nonnegative and self-adjoint operator $\Pi_M \in \mathcal{L}(V_n^0(\Omega))$, which belongs to $\mathcal{L}(V_n^0(\Omega), V_0^2(\Omega))$, solution to the following Riccati equation:

$$((A + A_{e,T})\Pi_M\xi|\zeta) + (\xi|(A + A_{e,T})\Pi_M\zeta) + (MB^*\Pi_M\xi|MB^*\Pi_M\zeta)_\Gamma = (\xi|\zeta), \quad (7.2)$$

for all $(\xi, \zeta) \in V_n^0(\Omega)\times V_n^0(\Omega)$. Thus, we introduce the linear operator $(\mathcal{D}(A_{\Pi_M}), A_{\Pi_M})$ in $V_n^0(\Omega)$ associated with Π_M:

$$\mathcal{D}(A_{\Pi_M}) = \big\{\xi \in V_n^0(\Omega) \mid A\xi + A_e\xi + BM^2(B^*\Pi_M)\xi \in V_n^0(\Omega)\big\},$$
$$A_{\Pi_M}y = A\xi + A_e\xi + BM^2(B^*\Pi_M)\xi,$$

and analogously as for Theorem 2 we prove that it obeys:

$$\mathcal{D}(A_{\Pi_M}^{s/2}) = \left\{ \xi \in V_n^s(\Omega) \mid \xi + PDM^2(B^*\Pi_M)\xi \in V_0^s(\Omega) \right\} \quad s \in [0, 2].$$

Moreover, we introduce the following norm $\|\cdot\|_{\Pi_M,s}$ on $\mathcal{D}(A_{\Pi_M}^{s/2})$:

$$\|\xi\|^2_{\Pi_M,s} = \langle \Pi_M^{(s)}\xi|\xi\rangle_{\mathcal{D}(A_\Pi^{s/2})',\mathcal{D}(A_{\Pi_M}^{s/2})} \text{ where } \Pi_M^{(s)} = A_{\Pi_M}^{*s/2+1/2}\,\Pi\, A_{\Pi_M}^{s/2+1/2}.$$

Finally, the following "localized" version of Theorem 5 can be proved with no additional difficulties.

THEOREM 12. *Let Π_M be the solution to (7.2), let $f \in \mathbf{H}^1(\Omega)$ and $v_b \in \mathbf{H}^{5/2}(\Gamma)$ be such that $\int_{\Gamma_j} v_b \cdot n = 0$, for all $j = 1\ldots N$, let $(z_e, r_e) \in V^3(\Omega) \times \mathcal{H}^2(\Omega)$ be a solution to (1.1), and let us consider the system:*

$$\partial_t z - \nu\Delta z + (z\cdot\nabla)z + \nabla r = f \text{ and } \nabla\cdot z = 0 \text{ in } Q, \ z(0) = z_0, \quad (7.3)$$
$$z = v_b + m^2\nu\partial_n\Pi_M P(z - z_e) + M^2(\psi n) \text{ on } \Sigma, \quad (7.4)$$
$$\Delta\psi = \nabla\cdot(\nabla z_e^T - z_e\cdot\nabla)\Pi_M P(z - z_e) \text{ in } Q, \quad (7.5)$$
$$\partial_n\psi = (-\nu\Delta + \nabla z_e^T - z_e\cdot\nabla)\Pi_M P(z - z_e)\cdot n \text{ on } \Sigma. \quad (7.6)$$

There exist $\rho > 0$ and $\mu > 0$ such that, if $\delta \in (0, \mu)$ and

$$P(z_0 - z_e) \in \mathcal{W}^s_{M,\delta} = \left\{ y \in \mathcal{D}(A_{\Pi_M}^{s/2}) \mid \|y\|_{\Pi_M,s} \le \rho\delta \right\},$$

then (7.3)-(7.6) admits a solution (z, r) in the set $\{(z_e, r_e)\} + \mathcal{D}^s_\delta$ given by (2.30). Moreover, the solution (z, r) is unique within the class of functions in

$$\big\{(z_e, r_e)\big\} + V_{loc}^{1+s,1/2+s/2}(Q) \times H_{loc}^{-1/2+s/2}(\mathcal{H}^s(\Omega)),$$

the mapping $t \mapsto \|P(z(t) - z_e)\|^2_{\Pi_M,s}$ decreases to 0, and there is $\sigma > 0$ such that:

$$\|P(z(t) - z_e)\|_{\Pi_M,s} \le \|P(z_0 - z_e)\|_{\Pi_M,s}e^{-\sigma t} \quad \forall t \ge 0.$$

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